MARICOPA MATHEMATICS MODULES

Preliminary Edition

Maricopa Mathematics Modules

Susan Riegle

Nazareth College

Mth 102: Thinking Mathematically

Houghton Mifflin Company
Boston New York

Senior Sponsoring Editor: Maureen O'Connor

Senior Associate Editor: Dawn M. Nuttall

Senior Project Editors: Cathy Brooks, Maria Morelli

Editorial Assistants: Amanda Bafaro, Cecilia Molinari

Senior Production/Design Coordinator: Sarah Ambrose

Director of Custom Publishing: Rusty W. Johnson

Marketing Manager: Ros Kane

Houghton Mifflin
Custom Publishing

Preface

Many mathematics faculty acknowledge that mathematics courses at the foundation level are too tightly packed with content that often seems disembodied. Students often fear this mathematics that seems irrelevant to their lives. Out of this realization, faculty in the Maricopa Community College District joined with their colleagues at local schools and at Arizona State University to rethink the curriculum. Over 75 faculty participated at one time or another, over a period of about five years.

We wanted the mathematics experience of students to be rich in mathematics, not shallow in symbol manipulation skills. Therefore, we selected mathematics content that would prepare students for their further studies in economics, biology, physics, accounting, etc., and not just the next mathematics course. We selected mathematics content that would prepare students for citizenship and lifelong learning. As each module was conceptualized, student outcomes for that module were developed first. Then the module was written, class-tested, refined, reviewed, and class-tested again.

Philosophy

Mathematics instruction at every level should propel students into future success as citizens, employees, employers, and as lifelong learners.

We believe that students need to gain a strong conceptual foundation of mathematics: that learning mathematics means to build strong connections among various topics of mathematics. We recognize that students do not build conceptual knowledge quickly, nor do they build a conceptual foundation by becoming proficient at doing template exercises and problems. Rather, students build conceptual knowledge by reflecting on their own knowledge and concepts.

To be successful in academic pursuits, as well as in life after college, students need to learn more mathematics than algebra. In the Maricopa Mathematics Modules, mathematical topics such as probability, sampling, geometry, and functions are interwoven with the development of algebra skills. In each module, students learn mathematics and mathematical language in a context of use. The mathematics in each module is grounded in the application of that mathematics. With the Maricopa Mathematics Modules, students do not wonder how this mathematics would ever be used.

The Maricopa Mathematics Modules

There are fifteen modules. These modules present mathematics to college students in new ways. The content is fresh; the pedagogical approach is engaging; the mathematical learning is powerful.

The modules can be configured into unique combinations to meet the needs of students in Foundation Mathematics (Arithmetic Review, Elementary and Intermediate Algebra), Quantitative Reasoning, Math for Liberal Arts, Technical Mathematics or short courses. They can form the entire course (see "Create Your Own Course"), or they can be used as "replacement units" with more traditional texts. Each of the fifteen modules is approximately equal to a one-credit course.

Data and Graphs

Students gather and analyze data from the fast food industry. They organize and describe this data in order to make and support decisions.

Geometry

Students use the Earth in order to explore and create visual models. Students define, measure, and work with the properties of 1-, 2-, and 3-dimensional objects.

Probability

Students interpret and calculate probabilities from frequency tables, pie charts, and experimental data and communicate results using the language of probability.

Linear Behavior

Students develop the tools to plot a data set, determine whether a linear relationship exists, and if so, describe it verbally, algebraically, and graphically.

Beat Ratios and Juggling Proportions

Students describe musical tempos, scheduling problems, and juggling using ratios and proportions.

Representations of Data

Students represent information in a variety of meaningful ways with appropriate justification.

Functions

Students learn the notation, concepts, and operations of functions, including transformations and inverses.

Sets and Logic

Students apply the rules of logic to solve problems in narrative, graphical, and numerical form. They develop critical thinking skills to solve problems in elementary set theory and survey analysis.

Systems

Students use symbolic, graphical, numeric and verbal representations of linear growth to solve problems in population growth and equilibrium analysis.

Sampling

Students investigate a variety of sampling schemes and historically significant data sets. They develop and apply sampling procedures to political surveys, manufacturing quality control, and the life sciences.

Exponential Growth and Decay

Students apply the distinguishing characteristics of exponential growth, in a variety of representations and problem situations.

Nonlinear Behavior

Students solve equations, interpret graphs, and utilize the fundamental concepts of functions with a variety of nonlinear functions.

Patterns

Students recognize and describe visual, numeric, geometric, and recursive patterns using appropriate language and notation.

Right Triangle Trigonometry

Students use a variety of instruments to determine the measures of angles and distances. Students apply right triangle models to applications.

Finance

Students explore the mathematics of finance including interest, amortization, annuities, credit cards, and lottery pay-offs.

Benefits

We believe that we should teach students what is most important. Therefore, we set our sights on six student outcomes that guide our work as authors and teachers. We call these the CREATE outcomes and believe that all students can CREATE mathematics.

Connect

Students employ a variety of methods (visual, symbolic, numeric, verbal, et al.) to represent and explore mathematical ideas. They construct and apply models that connect mathematics to the world. Students evaluate the soundness of their methodologies and results.

Reason

Students demonstrate clear reasoning in analyzing information. They develop and apply logical thinking skills to formulate and support conclusions.

Express

Students read, write, listen to, and speak mathematics both individually and in teams.

Appreciate

Students value the power of mathematics. They are confident, flexible, and persistent lifelong learners and users of mathematics.

Tap into Technology

Students discover the benefits and limitations of current technologies. They utilize technologies as resources for learning and problem solving.

Establish a Foundation

Foundation skills provide a basis for continual learning. Students acquire and develop a core of content-specific knowledge and abilities, as well as strategies for learning.

Pedagogical Blueprint

A module is broken down into several Lessons and uses the following effective features to help students achieve the CREATE outcomes.

Introduction Each lesson introduces the idea/concept to be studied. It lists the objectives and the materials needed.

Notes to the Instructor A page of notes for the instructor appears before every lesson. It provides the instructor with details about the lesson including: time estimates, materials needed, prerequisites, notes and Wrap-Up and homework information. There may also be margin notes throughout a module for instructor use. This information appears in the Instructor's Annotated Edition only.

Activities Each lesson is organized around student activities that are introduced by short expositions. Data is presented (or student-gathered), models are given (or student-created), and questions are asked which are specifically designed to lead students to a real-world, mathematical discovery/connection of the concept. Because the activities are part of the course material, it is easy to develop a classroom routine of active student involvement. Since students bring their own ideas and conjectures to the discussion, the whole class is enriched. Alternatively, some activities can be summarized by an interactive lecture; some activities can be done as homework. Most faculty teach the modules using a combination of lecture and collaborative groups.

Nitty Gritty Tied to a specific lesson, the Nitty Gritty feature appears intermittently and focuses on specific skills. The Nitty Gritty presents examples of a particular skill (as in solving an equation by taking the square root of both sides) followed by practice problems when it is important that students establish foundation skills.

Rule of 4 Students use four ways to express and explore mathematical ideas: graphically, numerically, symbolically, and verbally. In this way, students make a multitude of connections around specific mathematical concepts. For example, students may be asked to create a meaningful context for the function $P(x) = 1000 \cdot (1.05)^x$, such as, "a population of 1000 grows by 5% per year." They may then be asked to build a table of values for this function to show its doubling time, or to find the year when the population reaches 1750 by reading the graph or by solving an equation.

Writing Writing is an integrated feature. Students are asked to write explanations in their own words and express the real world meaning of something written mathematically. Writing can be found throughout all modules.

Technology Students use graphing calculators daily throughout all the modules. Any graphing utility can be used, but it should have data lists, data plotting and regression capability. The features of the TI-83 calculator fit the material nicely.

Graph Interpretation Problems These problems are integrated throughout and are closely related to writing as students see and explain how a real-world model and mathematical model relate.

Homework The homework problems are designed to reinforce and extend the concepts of the lesson. They provide opportunities for problem solving. In the homework, students make progress toward accomplishment of the CREATE outcomes, as well as skill development. Each problem has a unique contribution to make to a student's gain in skills and understanding.

Wrap-Up This lesson summary ask students to write their own summary encouraging them to be active participants in the learning process.

Assessment Most students appreciate the opportunity to show what they know in a variety of ways. For this reason we encourage a variety of assessments. In each lesson, students are encouraged to self-assess by completing the activities, the Wrap-Up boxes and the homework. Assessment ideas for instructors include quizzes, projects, research activities, and writing assignments as well as tests.

Glossary A glossary is provided for each module providing the student with defined key terms.

Selected Answers Selected answers to Activities, Nitty Gritties, and Homework exercises are provided to the students in the back of each module.

Create Your Own Course

A unique benefit of this project is that it allows every instructor to customize a course to his or her specifications by selecting modules on-line to create a textbook. The details of this selection process can be found at **www.hmco.com/college/mathematics**. A corresponding customized Instructor's Annotated Edition accompanies each order. Please note that each module is equal to about one semester credit hour; that is, a 3-credit course could expect to complete three modules.

As you review the modules for selection, be aware that our depth of coverage is quite different from conventional texts. For example, we may spend weeks developing the function concept and spend only one lesson exploring rational functions. We have recognized that not all competencies are created equal: building a few basic concepts provides a basis for skill development.

For your consideration, we have listed a possible grouping of modules for particular courses:

- **Arithmetic Review/Basic College Math/Prealgebra**
  Data and Graphs, Geometry, Probability, Linear Behavior, Beat Ratios
- **Elementary Algebra/Introductory Algebra/Beginning Algebra**
  Representing Data, Sets and Logic, Systems, Sampling, Functions
- **Intermediate Algebra**
  Exponential Growth and Decay, Nonlinear Behavior, Patterns, Right Triangle Trigonometry, Finance
- **Quantitative Reasoning or Liberal Arts**
  Data and Graphs, Geometry, Finance, Probability, Beat Ratios and Juggling Proportions, Exponential Growth and Decay, Representations of Data, Sets and Logic, Patterns, Sampling, and Functions.

Supplements

Instructor's Annotated Edition (IAE) This is an exact replica of the customized student text but also features margin annotations with teaching tips along with answers to the exercises. A page of notes to the instructor precedes every lesson. IAEs are ordered on-line with the student text.

Instructor's Resource Manual with Solutions Manual This useful guide contains detailed notes on specific modules, notes on managing this type of course, and test questions. The solutions are complete for all exercises. This supplement can accompany any customized text and should be ordered through your local Houghton Mifflin sales representative. (ISBN: 0-618-00053-4)

Student Solutions Manual This manual contains complete solutions to those exercises whose answers are found at the end of each module. This supplement can accompany any customized text and should be ordered through your local Houghton Mifflin sales representative. (ISBN: 0-618-00050-X)

Acknowledgments

The Maricopa Project wishes to thank the following for their unique contributions in bringing the Maricopa Project Modules to life, by their work in the Maricopa Mathematics Consortium (M^2C):

- Thanks to the M²C Curriculum Team, who took the risk to rethink what students need to learn in mathematics, who devised a curriculum framework, and who developed student outcomes for the Project. Many different mathematics faculty accomplished this over a period of four years, from 1994 through 1997.
- Thanks to the M²C Assessment and Evaluation Team, who developed procedures and forms for formative evaluation of the modules, who took the risk to rethink how to evaluate an entire curriculum.
- Thanks to the 22 M²C writers, who gave form and life to the curriculum framework by creating the initial drafts and many subsequent revisions of the course material modules.
- Thanks to those who have class-tested the modules.
- Thanks to both math and non-math faculty who have reviewed the modules.
- Thanks to Dr. Alfredo G. de los Santos, Jr., for his generous support of mathematics reform in the Maricopa College District, and especially for his unwavering support of the effort which has resulted in the Maricopa Project modules.

We would also like to give special thanks to the following people for their creative energy in developing the first drafts of the modules.

Beat Ratios and Juggling Proportions: Anne Dudley and Ron Epperlein
Data and Graphs: Teri Glaess, Barbara Herzog, and Keith Worth
Exponential Growth and Decay: Jim Krider and Alan Jacobs
Finance: Teri Glaess, David Dudley, Anne Dudley, and John Grima
Functions: Scott Adamson
Geometry: Teri Glaess and Barbra Samuels
Linear Behavior: Stephanie Peterson, Melinda Rudibaugh, and Howard Speier
Nonlinear Behavior: Karen Hay
Patterns: Scott Adamson, Karen Hay, and Kate Rozsa
Probability: David Dudley, Ron Epperlein, and Sharon Walker
Representations of Data: Paula Cheslik and Kyle Kirkman
Right Triangle Trigonometry: Teri Glaess, Donna Tannehill, and Keith Worth
Sampling: David Dudley and Teri Glaess
Sets and Logic: Scott Adamson and David Platt
Systems: Ron Epperlein, Jim Krider, and Sharon Walker

Partial support for this work was provided by the National Science Foundation's Advanced Technological Education (ATE) Program through grant #DUE9602386. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect views of the National Science Foundation.

Contents

Module

Patterns

Scott Adamson

Paula Cheslik

Anne Dudley

David Dudley

Teri Glaess

Karen Hay

Alan Jacobs

Keith Worth

LESSON 1: Introducing Patterns

Introduction

Every day of our lives, we observe, use, and create a variety of patterns. In the morning, we walk across the tile pattern on the bathroom floor. In school, children use patterns to trace pictures or make shapes. At work, we analyze data to see if there is a pattern. Patterns can be visual, numerical, verbal, natural, human-made or in many other forms.

Learning Objectives
In this lesson, you will . . .

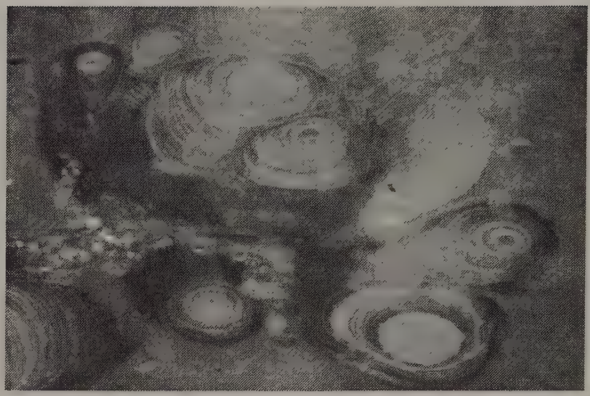
- recognize and classify patterns
- analyze the meaning of the word *pattern*
- develop a definition of the word *pattern*

Introducing Patterns

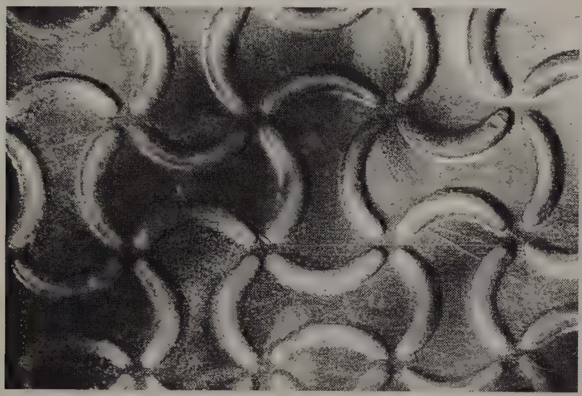
We hear the word *pattern* every day: sleep patterns, floor patterns, bad patterns of behavior, natural patterns, fabric patterns, geometric patterns, and so on. We see patterns all around us in the natural world of plants and animals and in different things that people make. But just what do we mean when we say there is a pattern? Does the concept of pattern involve only one thing, or are there many different kinds or categories of patterns? We will start to explore all these different patterns by trying to recognize them in pictures.

To begin, study the following pages, which contain pictures of a variety of patterns found in the world around you. Try to identify each picture. Ask yourself, "What is it?" Record your answers.

1. ______________________________

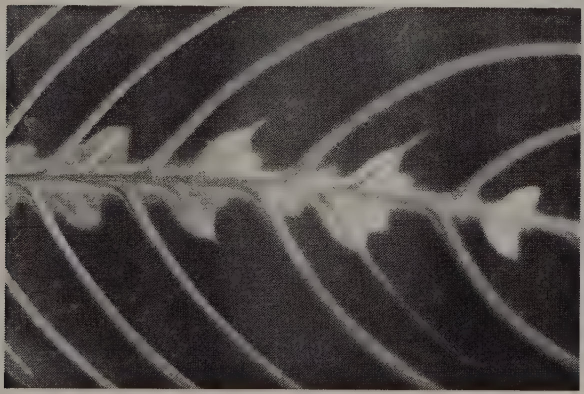


2. ______________________________



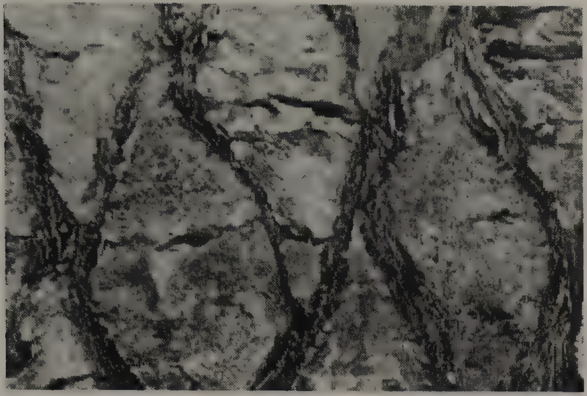
3. ______________________________

4. ______________________________

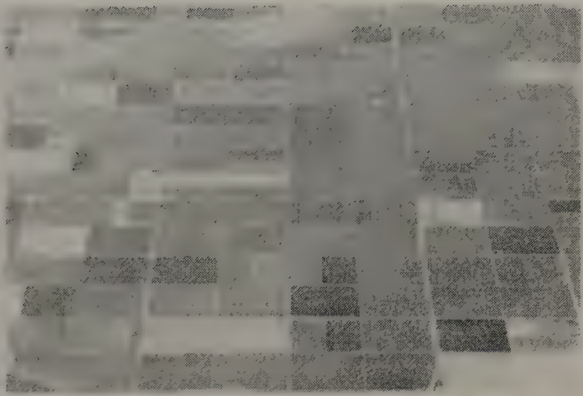


5. ______________________________

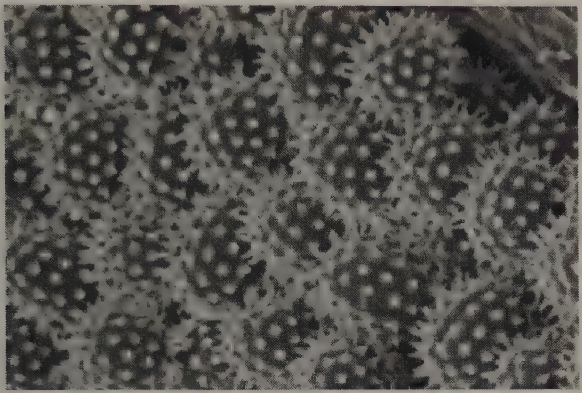
6. ______________________________



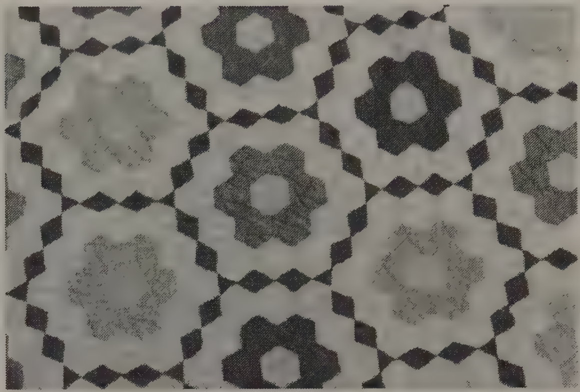
7. ____________________



8. ____________________

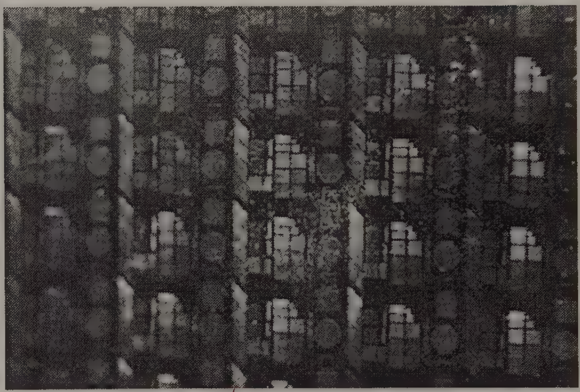


9. ____________________

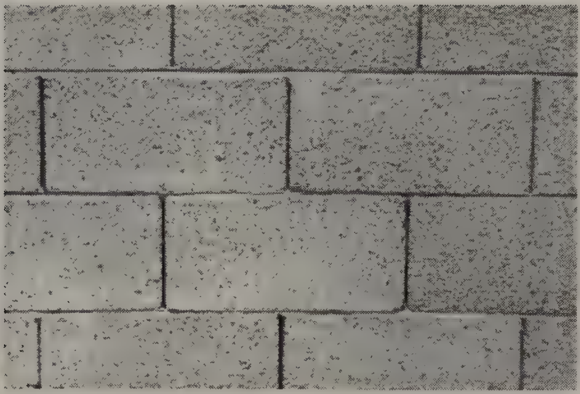


10. ____________________

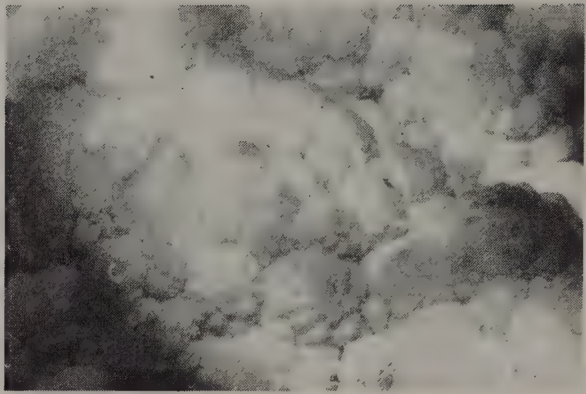
11. ____________________



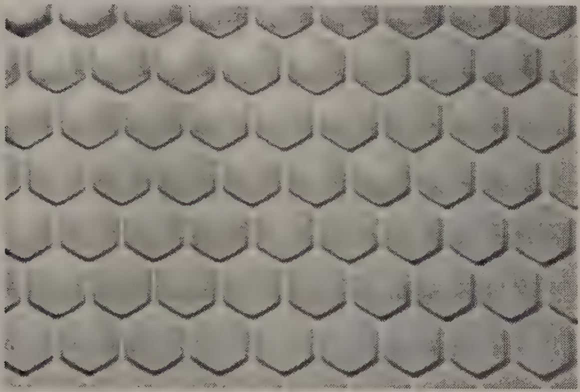
12. ____________________



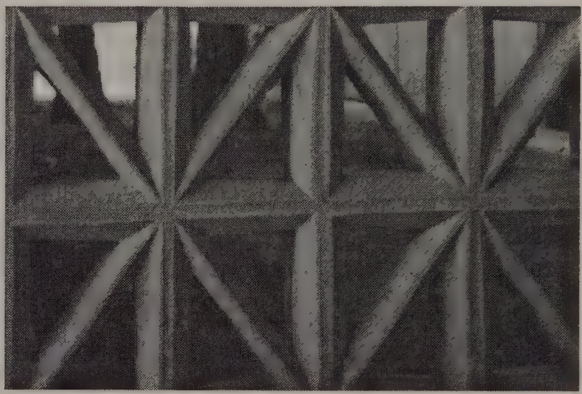
13. ______________________



14. ______________________



15. ______________________



16. ______________________

17. ______________________

18. ______________________

From the pictures, explain in a few sentences what you mean when you say, "There is a pattern here." Record your answers.

Below is a list of pattern categories. Complete the table by writing the number of each picture in the appropriate box. It is possible for a picture to belong to more than one category.

Pattern Type	**Which Pictures Belong to Each Category?**
Natural patterns	
Human-made patterns	
Art patterns	
Mathematical patterns	
Ring patterns	
Branching patterns	
Spiral patterns	

Table 1 Classifying Patterns

Read and consider the definitions of *pattern* that follow. Consider whether you agree or disagree with each.

- A pattern is the repetition of an object or an operation.
- A pattern is a decorative design of shapes or forms.
- A pattern is anything designed to serve as a model or guide for something to be made.
- A pattern is a uniform design.

Develop your own definition of *pattern* and write it here.

WRAP-UP

Every day we see patterns all around us. Look for them.

List any patterns you see around you now.

Project introduction

At the end of this module, you are to present a collection of patterns. The patterns can be of any type (natural, human, art, mathematical, and so on) and should revolve around a single theme (such as sports, architecture, plants, cars, people, food, or any other).

Begin collecting examples of patterns and think about how you will present them.

HOMEWORK

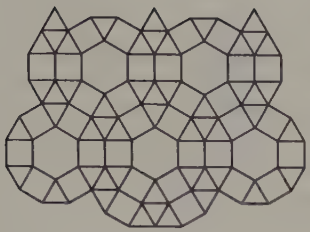
1. Find at least four different patterns that you like in different places. Record your findings in the table below.

What Is the Object?	Where Did You See It?	Why Do You Like It?	What Is the Purpose of the Pattern?

Table 2 Patterns in the World

2. Do a search on the World Wide Web to see what you find about patterns.

Lesson 2: Introduction to Tessellations



Introduction

The tile pattern that covers the floor of your bathroom or kitchen is called a tessellation. The pattern formed by the bricks of a fireplace is called a tessellation. In addition to being beautiful, strong, and functional, tessellations illustrate useful mathematical concepts, including regular and irregular shapes, polygons, angle, angle measure, angle sum, vertex, reflection, translation, and rotation. In this lesson, you will learn about polygons and how some of them can be used to create tessellations. You will also create your own tessellations.

Learning Objectives

In this lesson, you will . . .

- distinguish between regular and irregular polygons
- identify and draw 3-sided to 8-sided polygons
- create and analyze tessellations created from regular polygons

Materials

- tracing paper or copies of the regular polygons
- scissors
- construction paper
- pencils, erasers, colored pencils or markers

Introduction to Tessellations

> A *tessellation* is a tiling made up of copies of a shape or shapes that fit together with no gaps or overlaps. The word tessellation comes from the Latin word *tessela* - the small square tiles used in ancient Roman mosaics

Tessellations have appeared in many cultures throughout history. In the pioneer days of the United States, women pieced together scraps of cloth to make beautiful and functional quilts. Brick masons arrange bricks to build strong walls and to form appealing designs. Farmers partition their land to maximize production and provide good access. Engineers design interlocking cement blocks for strength and flexibility in construction. Kitchen and bathroom countertops and floors are often tessellations made of linoleum or ceramic tile. Artists form mosaics and collages to decorate floors and walls. Bits of colored glass are pieced together to form beautiful stained glass windows.

The building blocks of the tessellations are called **tiles.** Many tessellations are constructed from polygons, particularly regular polygons. A **regular polygon** is made up of sides of equal length and has equal angles. Examples of regular polygons include the equilateral triangle and square. Other polygons, such as the pentagon, hexagon, heptagon, and octagon, may be regular or not.

✎ The table on the facing page shows sketches of several polygons. Complete the table by studying the diagrams of the regular polygons and determining the number of sides of each. Fill in the name of those figures you know, and try searching a dictionary or other resource for those you do not know.

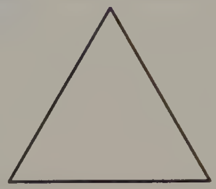
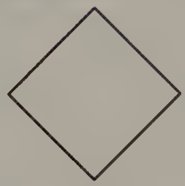
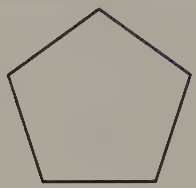
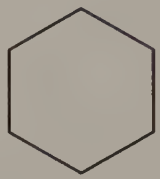
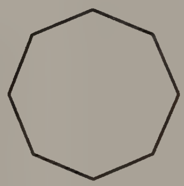
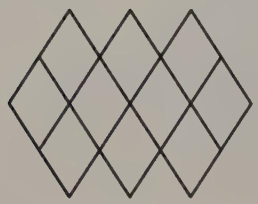
Number of Sides	Name	Diagram
		
		
		
		
		
		

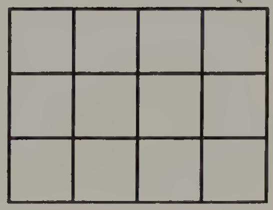
Table 3 Polygons

Several examples of tessellations appear below. Some of the tessellations were constructed from tiles that are regular polygons; some were not. (Remember, a regular polygon is a polygon with all angles the same size and all sides the same size.) Write "yes" below the pictures formed using regular polygons, and write "no" for the others. If your answer is no, explain why.

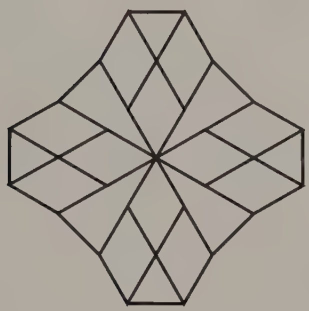
1.



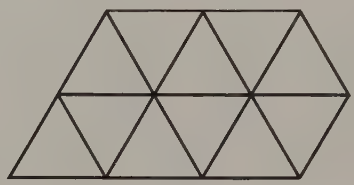
2.



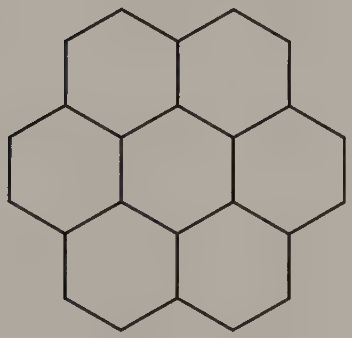
3.



4.



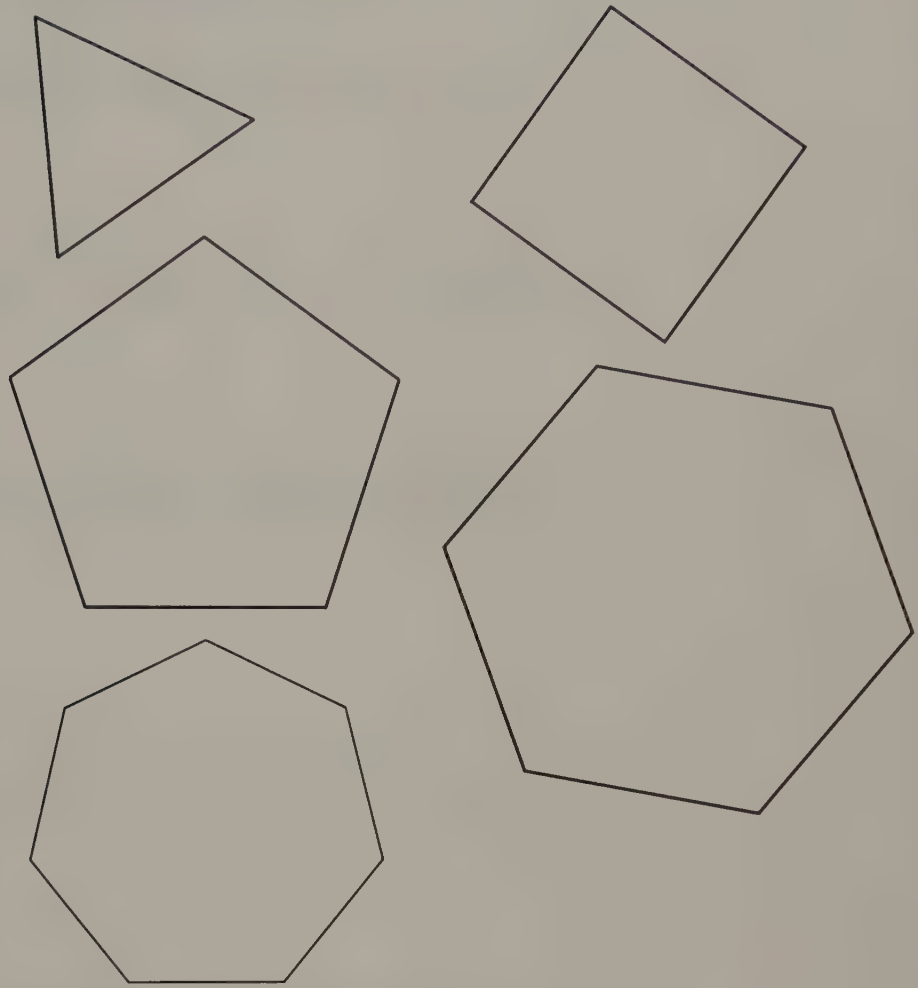
5.



6.

ACTIVITY 1—CREATING TESSELLATIONS WITH REGULAR POLYGONS

1. Carefully trace the regular polygons below to construction paper, and cut them out. These will be your tile patterns.



2. Use your tile patterns to cut eight more copies of each tile.

3. Try to construct tessellations using only triangles. Try again using only squares, then pentagons, hexagons, etc. Draw a sketch of each successful tessellation.

4. Were you able to construct a tessellation with each of the tiles?

5. For which of the polygons were you able to construct more than one tessellation? Give examples.

6. Try to construct tessellations using several different tiles (that is, a mix of triangles, squares, pentagons etc.). Draw a sketch of each successful tessellation.

7. Were you able to construct more than one tessellation using the same combination of regular polygons? Give examples.

8. Some regular polygons and combinations of regular polygons tessellated successfully; others did not. Why do you think some regular polygons were successful and others were not?

WRAP-UP

A tessellation is a tiling made up of copies of a shape or shapes that fit together with no gaps or overlaps. Many tessellations are made from regular polygons.

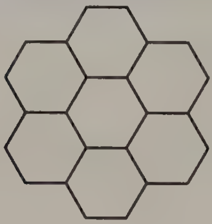
1. What is a regular polygon?

2. Some polygons will not tessellate. Explain why.

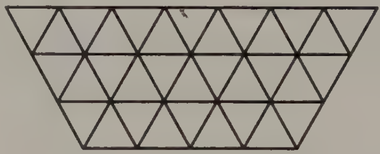
HOMEWORK

1. Study the pictures below. Determine whether the tessellation is formed with regular polygons. Explain.

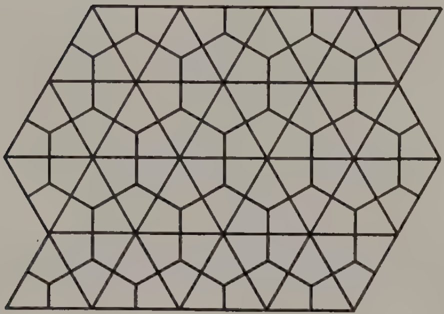
a.



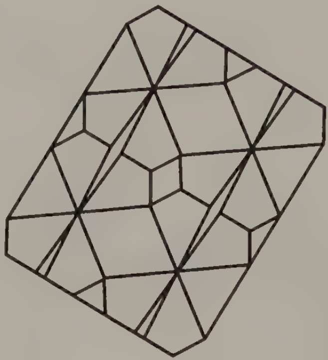
b.



c.



d.



2. If you were allowed to use only one type of tile, indicate which of the following regular polygons could be used alone to successfully tessellate. Explain why you believe some regular polygons tessellate successfully and others do not.

Polygon	**Tessellate?**	**Explain**
Triangle		
Square		
Pentagon		
Hexagon		
Heptagon		
Octagon		

Table 4 Tessellation Summary

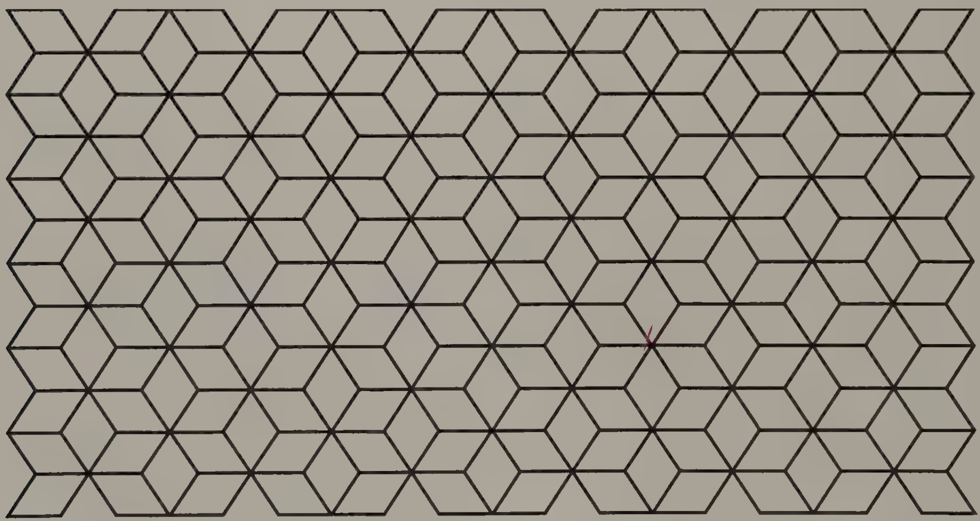
3. Look for and collect at least three examples of tessellations in the world around you, and thoroughly describe each tessellation.

Sketch	Where Did You Find It?	What Is Its Purpose?	Polygons Used? Regular?

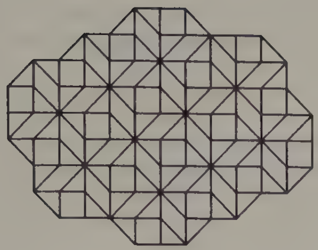
Table 5 Tessellations in the World

4. The tessellation below is a polygonal tessellation, but the polygons are not regular because the angles are not equal. The polygon in the tessellation below is called a rhombus. Looking at or coloring this tessellation in one way produces a three-dimensional effect; it appears to be a stack of building blocks or stairs. Looking at or coloring the tessellation in another way produces a star-quilt effect.

   Use crayons or colored pencils to color different parts of the tessellation. Make one part look like a stack of building blocks. Make the other part look like a star-quilt.



LESSON 3: Angling for Patterns



Introduction

In the previous lesson, you found that some of the tiles were not able to tessellate. What was the reason? It might be nice to know if and how the ceramic tiles you purchased will fit together for your bathroom floor before you get out the cement. In this lesson, you will more closely analyze the characteristics of regular polygons in order to determine whether they can tessellate.

Learning Objectives

In this lesson, you will . . .

- use a protractor to measure angles
- develop the angle sum formula for polygons
- analyze the relationship between angles of regular polygons and suitability for tessellating

Materials

- protractor

Angling for Patterns

In this lesson we will develop a systematic method for determining which regular polygons and combinations of regular polygons will successfully tessellate. You will need a protractor.

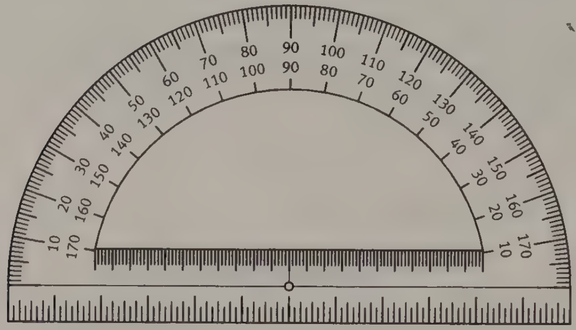


Figure 1

A protractor is a device used to measure angles. The unit of measure of an angle is the degree(°). A **right angle** has a measure of 90°, and a straight line has a measure of 180°. To use a protractor to measure an angle, place the center of the base of the protractor over the vertex of the angle. The line along the base of the protractor should coincide with one of the sides of the angle to be measured. To read the degree measure of the angle, read the number through which the other side of the angle passes on the curved part of the protractor.

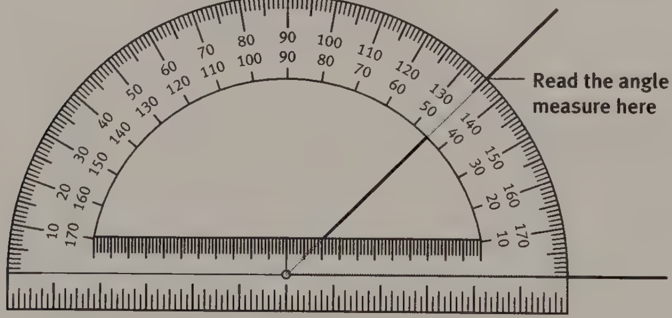
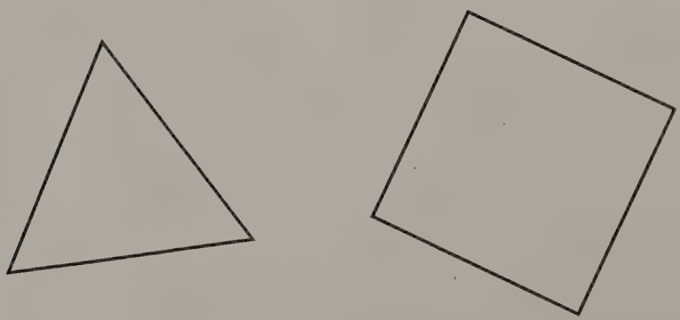
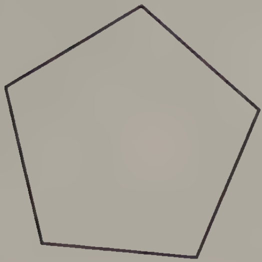
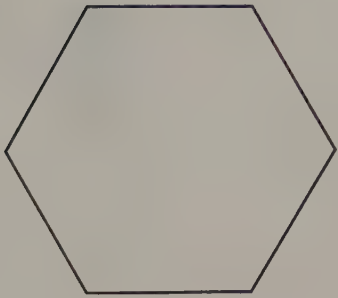


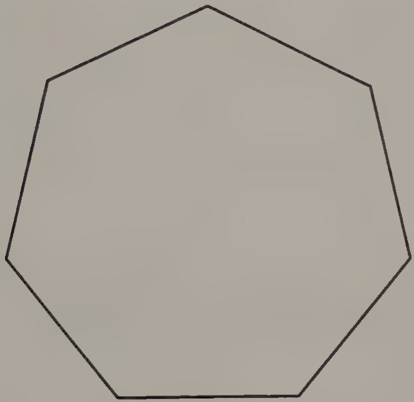
Figure 2

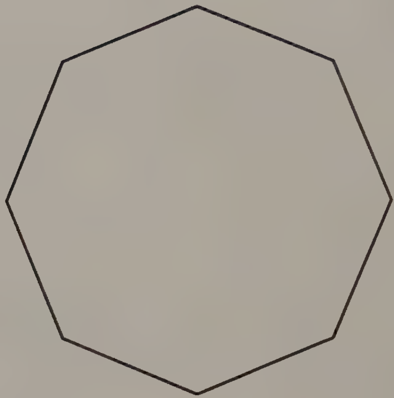
Use your protractor to measure each of the angles of the regular polygons pictured and complete the table. The table asks for the angle measure and the angle sum. The angle measure, m, is the measure of a single angle of the polygon. The angle sum, s, is the sum of all the angle measures of a polygon. For example, a square has four 90° angles, so $m = 90°$ and $s = 90 + 90 + 90 + 90 = 360°$.











Name	**Number of Sides, *n***	**Angle Measure, *m***	**Angle Sum of the Polygon, *s***
Triangle			
Square			
Pentagon			
Hexagon			
Heptagon			
Octagon			

Table 6 Angle Sums of Regular Polygons

Study your completed table. You may observe a relationship between the angle sum of the polygon and the number of sides. In the following activity, you will find a series of questions and steps designed to help you find this pattern and use it to write a formula for the angle sum of a polygon. Use a spare piece of paper to cover up the questions and steps until you are ready to see them.

ACTIVITY 1—FINDING A FORMULA FOR THE ANGLE SUM OF A POLYGON

Try to determine the formula on your own by studying the preceding table carefully. Consider how the angle sums change from one polygon to the next. If you get stuck, allow yourself to look at one question. After answering a question, try again to construct the formula before looking at the next question.

1. What is the difference between the angle sum of the equilateral triangle and the angle sum of the square?

2. What is the difference between the angle sum of the square and the angle sum of the pentagon?

3. What is the difference between the angle sum of the pentagon and the angle sum of the hexagon?

4. In general, what is the difference between the angle sums of two polygons when one polygon has exactly one more side than the other?

5. Write the angle sum of each polygon as a product of some number and 180°. Record your answers in the fourth column table.

6. Compare the number multiplied by 180° with the number of sides that the polygon has. What is the difference between the number multiplied by 180° and the number of sides?

Name	**Number of Sides, *n***	**Angle Sum, *s***	**Angle Sum Expressed as a Product**	**Angle Sum Expressed in Terms of *n* and 180°**
Triangle	3	180°	$1 \cdot 180$	$(3 - 2) \cdot 180$
Square				
Pentagon				
Hexagon				
Heptagon				
Octagon				

Table 7 Angle Sum

7. Write the angle sum of each polygon as a product of an expression involving the number of sides and 180. Record your answers in the fifth column of the table.

8. Let s = the angle sum of a polygon with n sides. Write a formula expressing s in terms of n.

Formula for the angle sum of a polygon

The angle sum formula can be used to tell us the sum of the angle measures for any polygon. For example, the formula can be used to find the angle sum of a 10-sided polygon (a decagon).

$$\begin{aligned} s &= (10 - 2) \cdot 180° \\ &= 8 \cdot 180° \\ &= 1440° \end{aligned}$$

The angle sum of a decagon is 1440°.

When creating tessellations, it is useful to know the angle measure of a single angle in a polygon. Recall that the angles of regular polygons are all equal.

How could we use this fact to modify the angle sum formula and construct an angle measure formula for a single angle in a regular polygon?

Angle measure formula for regular polygons

Write the formula here.

The angle measure formula can now be used to determine the angle measure of an angle in a 12-sided regular polygon (a dodecagon).

$$\begin{aligned} m &= \frac{(12 - 2) \cdot 180°}{12} \\ &= \frac{10 \cdot 180}{12} \\ &= \frac{1800}{12} = 150° \end{aligned}$$

The angle measure of any angle in a regular dodecagon is 150°.

ACTIVITY 2—FINDING THE ANGLE SUM OF A TESSELLATION VERTEX

The concepts of angle measure and angle sum can now be used to help determine which combinations of regular polygons will successfully tessellate and which will not. Answer each of the following questions or perform the specified tasks. Some of the questions will require careful thought. Do not rush.

1. Study your tessellation sketches from Lesson 2 and refer to the table of angle measures or use the angle measure formula to determine the angle measure of each polygon in the tessellation. Then sum the measures of all the angles at any vertex in each of your tessellations.

2. Did you find the same vertex angle sum for all your tessellations?

3. Explain why you think the vertex angle sums should or should not all be the same.

4. What is the sum of the angle measures of all the angles at any vertex in a tessellation?

5. How could you use the vertex angle sum of a tessellation and the list of regular polygon angle measures to select a polygon that will successfully tessellate
   a. with itself?

   b. with other regular polygons?

6. Sketch an example of at least three of the possible tessellations.

WRAP-UP

1. The angle sum, *s*, can be calculated by:

2. For regular polygons, the angle measure, *m*, can be calculated by:

3. Some combinations of regular polygons tessellate successfully; other combinations do not. Why?

HOMEWORK

1. Complete the table. You may need a dictionary to determine the number of sides.

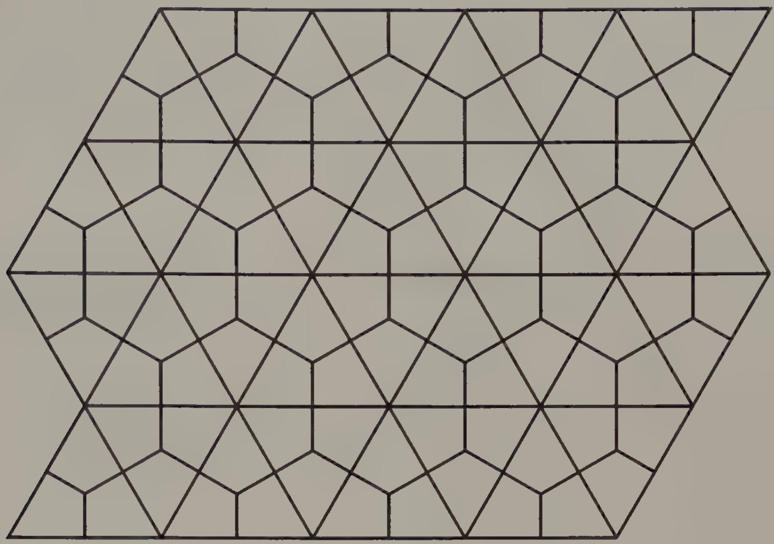
Name	Number of Sides, *n*	Angle Measure, *m*	Angle Sum of the Polygon, *s*
Nonagon			
Decagon			
Dodecagon			
Pentadecagon			
Icosagon			

Table 8 Angle Sum of Larger Regular Polygons

2. Sketch or describe three possible regular polygon tessellations. Include at least one of the regular polygons from Problem 1 in each tesselation.

3. There are a number of sites on the World Wide Web that provide information about tessellations. Do a keyword search on tessellation and write a couple of sentences communicating what you found.

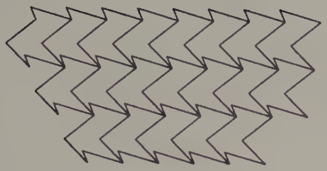
4. Another way to create polygonal tessellations is to overlay two or more simple polygonal tessellations. Perhaps the most famous tessellation of this type is called the *Chinese Lattice.* It is constructed by overlaying triangles and hexagons.



To construct the overlay, you can draw the triangle tessellation on a blank overhead transparency or tracing paper and lay it on top of the hexagon tessellation. Alternatively, you can draw points in the center of the hexagons and join the points to form the overlaying triangles.

Create at least two other tessellations of this type by overlaying a variety of simple tessellations.

LESSON 4: Translating the Language of Tessellations



Introduction

In Lesson 2, you created tessellations with regular polygons. How did you do it? Did you repeatedly shift the tile to the right? Did you repeatedly rotate your tile? Did you repeatedly flip your tile over? Was it a combination of these? In order to create a tesselation, you performed a rigid motion with your tile.

Learning Objectives

In this lesson, you will . . .

- utilize the language of transformations (translation, reflection, rotation, and glide reflection) to analyze and create tiles and tessellations

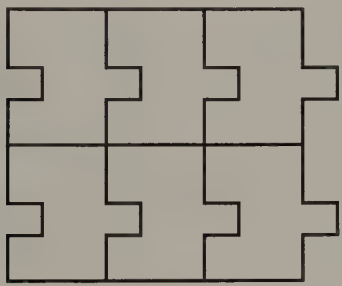
Materials

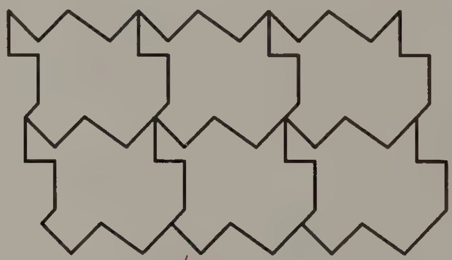
- index cards
- scissors
- paper

Translating the Language of Tessellations

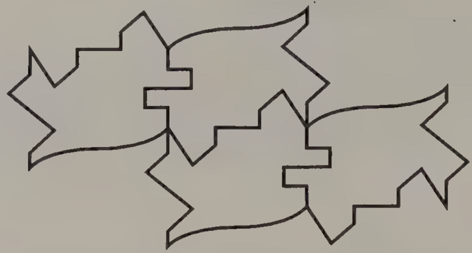
In mathematics, a rigid motion is a transformation that leaves an object's size unchanged. Tessellations are created by repeatedly performing one of the following rigid motions with a tile: translation, reflection, rotation, or glide reflection.

A **translation** is a shift of a specified direction and distance. A tile can be copied and shifted up, down, left, or right to create a tessellation.





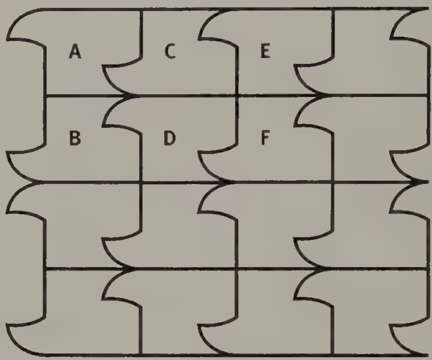
A **rotation** is a rigid motion that turns the image around a center with a specified angle of rotation.



A **reflection** is a rigid motion that creates a mirror image and is specified by a line of reflection (that is, the location of the mirror). A **glide reflection** is a combination of a reflection and a translation along the line of reflection.

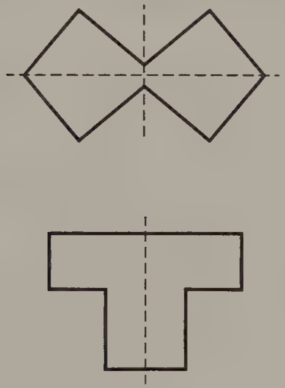
In the figure at the right, B is a reflection of A. D is a reflection of C.

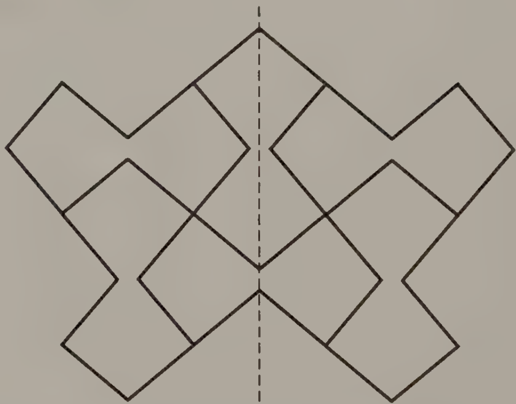
F is a glide reflection of A. A is first glided (translated) to E and then reflected to F.



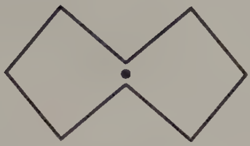
An individual tile or an entire tessellation may exhibit some of these same characteristics.

A tile or tessellation is said to have **reflection symmetry** if the tile or tessellation can be folded in half along some midline and all edges match up.

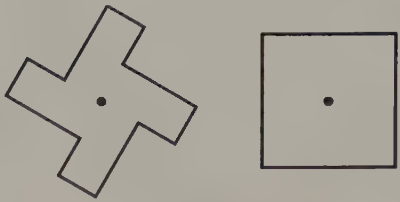




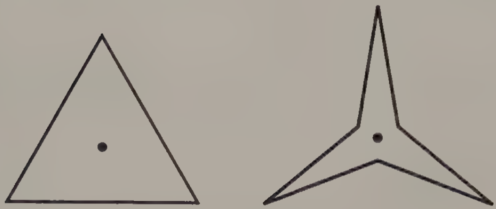
A tile or tessellation is said to have **rotation symmetry** if the tile or tessellation can be rotated a specified angle onto itself. Rotations are often classified as 2-fold, 3-fold, 4-fold, and 6-fold rotations. A 2-fold rotation allows a tile to be rotated twice (halfway around each time) before returning to its starting position such that after each rotation, the tile looks unchanged. A 3-fold rotation allows a tile to be rotated three times (one-third of the way around each time) before returning to its starting position such that after each rotation, the tile looks unchanged. And so on.



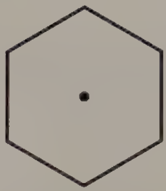
2-fold
rotational symmetry



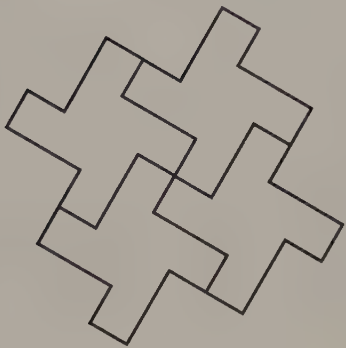
4-fold
rotational symmetry



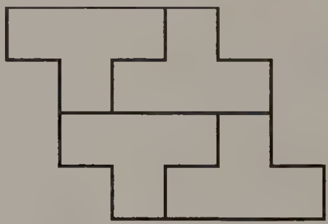
3-fold
rotational symmetry



6-fold
rotational symmetry



4-fold
rotational symmetry

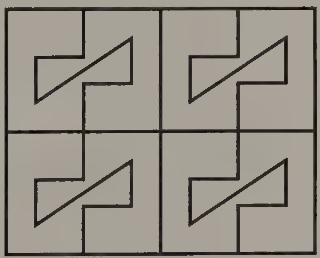


2-fold
rotational symmetry

ACTIVITY 1—IDENTIFYING SYMMETRY

Study the tessellations below and determine whether each tile and/or tessellation exhibits reflection or rotation symmetry.

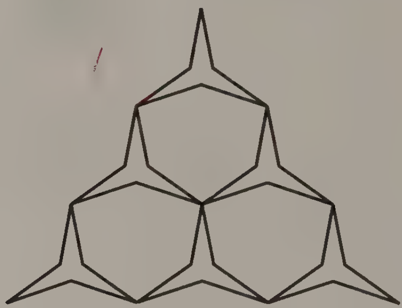
1.



Tile:

Tessellation:

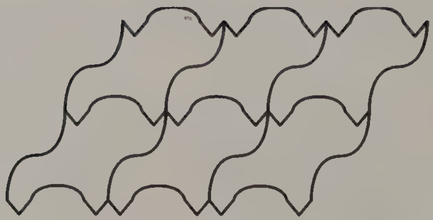
2.



Tile:

Tessellation:

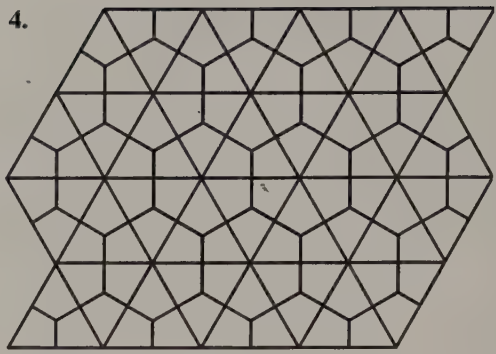
3.



Tile:

Tessellation:

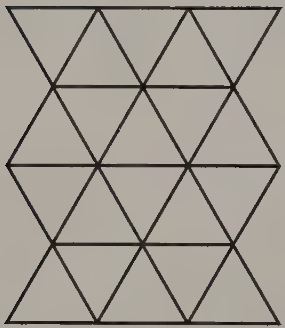
4.



Tile:

Tessellation:

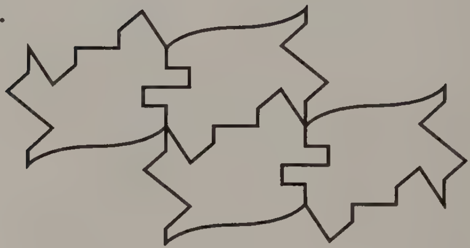
5.



Tile:

Tessellation:

6.



Tile:

Tessellation:

ACTIVITY 2—CREATING TILES USING TRANSLATION

You will need 3×5 index cards, cut into 3×3 squares. Study the diagrams and follow the instructions below to construct tessellating tiles.

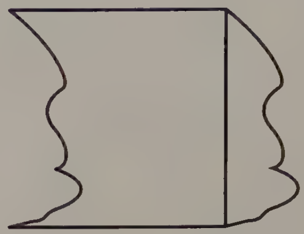
We are going to create tiles by translating one side of the tile to form the opposite side.

Instruction **Diagram**

1. Take one index card and draw a curve or squiggly line along one edge.

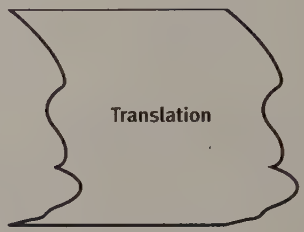
2. Cut out along the curve.

3. Translate the cut piece to the opposite side of the tile.



4. Use tape to attach the cutout.

5. Write the word *translation* on your tile.



You are now holding a tile that will tessellate by translating.

ACTIVITY 3—CREATING TILES USING REFLECTION

The sides of a tile with reflection symmetry must themselves have 180° rotational symmetry to ensure the ability to tessellate.

Instruction **Diagram**

1. Take one index card and draw a curve or squiggly line along half of one edge.

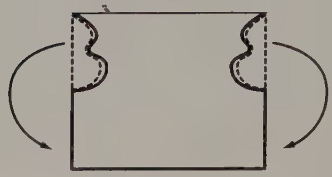
2. Cut out the curve. (Keep this cutout.)

3. Fold your card in half so that the cut side lines up with its opposite side.

4. Trace along the curve.

5. Unfold the card and cut along the curve. (Keep this cutout.)

6. Place the card on the desk in front of you with each cutout in its appropriate place.



7. Rotate each cutout 180° so that it matches up with the card and secure with tape.

8. Write the word *reflection* on your tile.

You are now holding a tile that exhibits reflection symmetry. It will tessellate by glide reflection.

Use your tiles to create tessellations. Place the tile on a piece of paper and trace around it. Reposition the tile so that it interlocks with the pieces already drawn and trace around it again. Repeat this process until you have a reasonable-sized tessellation. A single tile may be used to create several different tessellations.

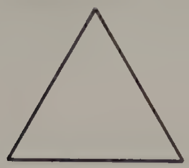
WRAP-UP

Tessellations and tiles exhibit a number of interesting rigid motions or symmetries, including translation, reflection, rotation, and glide reflection. Describe each of these motions.

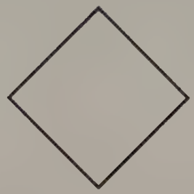
HOMEWORK

1. *Symmetries in Polygonal Tiles.* Polygonal tiles (see Lessons 2 and 3) typically illustrate a high number of symmetries. Study the polygons below and list all the symmetries you can determine.

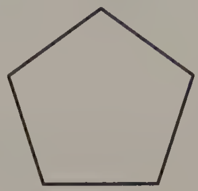
a.



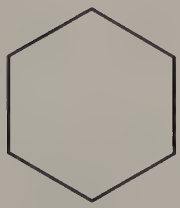
b.



c.

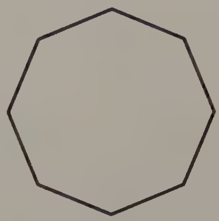


d.



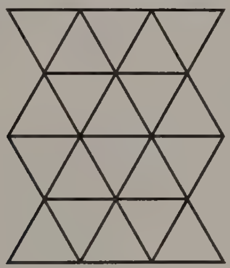
e.

f.

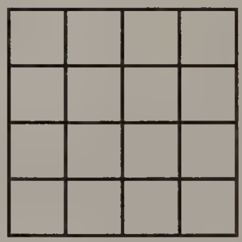


2. *Symmetries in Polygonal Tesselations.* Polygonal tessellations typically illustrate a high number of symmetries. Study the polygonal tessellations below and list all the symmetries you can determine.

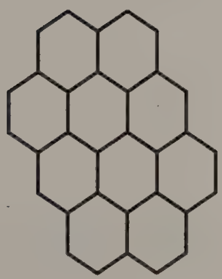
a.



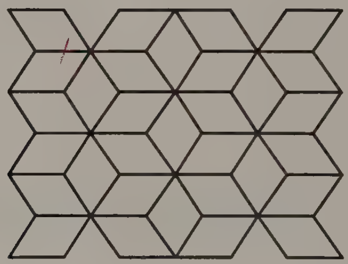
b.



c.



d.



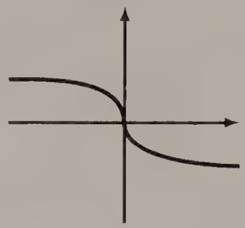
3. What is the relationship between the symmetries of a tile and the symmetries of a tessellation in which the tile appears?

4. Glide reflection is really just a combination of two of the other types of rigid motions. Which two?

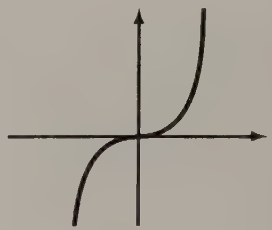
5. *Symmetry in Graphing.* Symmetry is sometimes evident in the graphs of functions. Study the graphs below and list all the symmetries that you see.

a.

b.



c.



d.

e.

6. *Symmetry in Nature.* Symmetry often appears in nature. Human beings possess reflection symmetry. Starfish possess 5-fold rotational symmetry and reflection symmetry. Find at least three other examples of symmetry in nature. For each example, provide a sketch, as well as a list and explanation of all the symmetries possessed by the example.

Sketch of Object	**Symmetries**

Table 9 Symmetries of Objects in Nature

7. *Symmetry in Words.* Many letters of the alphabet (and even some words) exhibit symmetry. For example, a capital A has reflection symmetry, and the letter S has 2-fold rotational symmetry. The word *wow* has reflection symmetry. Give a few examples of letters and/or words that exhibit symmetry.

EXTENSIONS

1. Tiles Based on Triangles and Hexagons. Tiles made in this lesson were based on a square. It is also possible to make tiles based on triangles and hexagons. Construct three tiles based on triangles and three tiles based on hexagons. One tile of each base shape should illustrate each of the following symmetries: translation, rotation, and glide reflection.

2. Tessellation-based Games. A number of games are based on tessellating a plane. Two such games are Tanagrams and Polyominoes. In each game, a player is given a set of tiles that must be arranged within an outline so that there are no gaps or overlaps and the entire outline is filled in. Find a copy of either or both games and see how many shapes you can form.

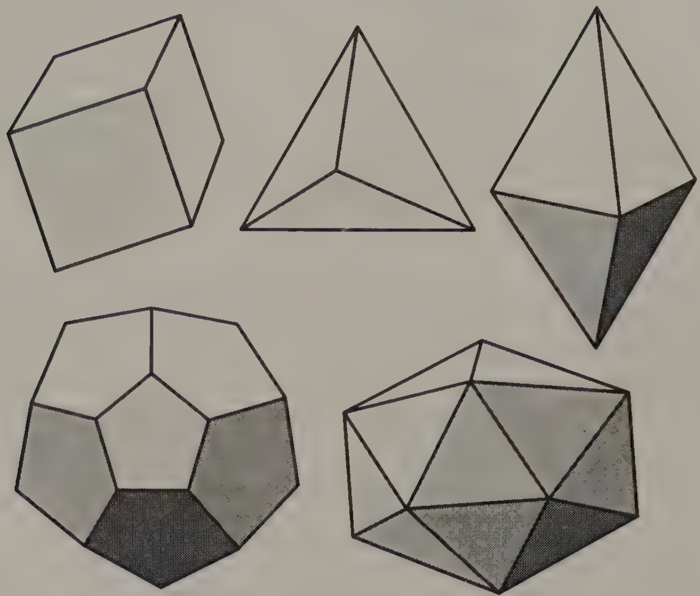
3. Art. M. C. Escher was a famous artist who specialized in fascinating tessellations. His tessellations possess many of the transformations discussed in these lessons. Locate copies of a number of Escher's tessellations and identify the symmetries they possess.

   Study the tiles you made. Do they remind you of anything? Try coloring or decorating your tiles so that they look like animals, machinery, or plants. Using copies of your decorated tiles (a photocopy machine might be helpful), construct a tessellation similar to an Escher tessellation.

4. Go to the library or the World Wide Web and research one of the following. Write a one-page paper describing the type of pattern, how it could be viewed as a tessellation, and its history.

Penrose tessellation	Chinese lattice
Tamil drawings	Shongo sand patterns
Celtic knots	Taj Mahal floor tilings

LESSON 5: Introduction to Polyhedra



Introduction

So far, we have looked only at polygons and tessellations in two dimensions, on a plane. When polygons are put together to create three-dimensional objects, they are called polyhedra. In this lesson, you will apply what you have learned about regular polygons and tessellations to learn about polyhedra.

Learning Objectives
In this lesson, you will . . .

- distinguish between regular and irregular polyhedra
- identify the parts and names of polyhedra
- learn some historical facts about the Platonic solids

Introduction to Polyhedra

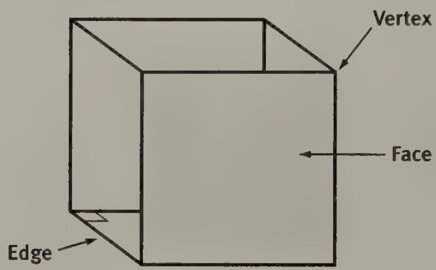
Polyhedra are three-dimensional objects constructed from polygons. The word *polyhedra* comes from two Greek words: *poly,* which means "many"; and *hedra,* which means "seats." Hence, a polyhedron is a three-dimensional object capable of being seated on any of its faces.

Many human-made and natural objects have polyhedral shape. Books, stereo speakers, box springs, and boxes are shaped like rectangular solids; tents, roofs, and doorstops are shaped like prisms or pyramids; buildings are shaped like pyramids, tetrahedra, rectangular solids, and even pentagonal solids. A number of small sea creatures and chemical compounds are shaped or organized like icosahedra, tetrahedra, and dodecahedra. Common table salt crystals have the shape of a cube. Chromite, iron, gold, and diamond occur in octahedral crystals, and quartz occurs in a variety of crystal forms.

In addition to appearing in human-made and natural forms, polyhedra are beautiful and intriguing. They appear frequently in design, symbol systems, and games. They also illustrate useful mathematical concepts, including angles, symmetry, vertex, edge, face, surface area, and volume.

A polyhedron constructed from a single type of regular polygon (such as all triangles or all pentagons) is known as a **regular polyhedron.** Recall that a regular polygon has sides of equal length and angles of equal size. Examples of regular polygons include the equilateral triangle, square, regular pentagon, and regular hexagon.

A cube is a regular polyhedron constructed from six squares.

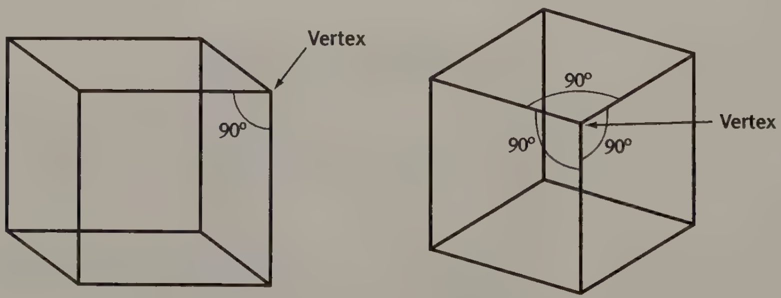


A point at which the corners of the squares meet is called a **vertex,** the surface of each square is called a **face,** and the line segment where the sides of two squares meet is called an **edge.**

How many regular polyhedra are there?

We can determine exactly how many polyhedra are possible by referring to our study of tessellations. Recall that the angle sum of a tessellation vertex is 360°. In other words, when the angle sum of a vertex is 360°, the tessellation is two-dimensional: flat. To create a three-dimensional tessellation (like a soccer ball), the angle sum of each vertex must be less than 360°. A vertex with an angle sum less than 360° forms a corner in a polyhedron.

For example, the cube is a regular polyhedron constructed from six squares. Three squares meet at each vertex. The measure of an angle of a square is 90°; therefore, the angle sum of a vertex of a cube is the sum of the angle measures of the three squares meeting at the vertex; $90° + 90° + 90° = 270°$.



Thus a regular polyhedron (a three-dimensional tessellation of regular polygons) can be constructed by selecting a single regular polygon and using a number of copies to form a vertex of the polyhedron such that the angle sum is less than 360°.

How many faces are needed to make a vertex? Is there a minimum number? A maximum number?

ACTIVITY 1—HOW MANY REGULAR POLYHEDRA ARE THERE?

1. How many regular polygons can be joined at a vertex so that the sum of the angle measures is less than 360° and the formation also gives a polyhedron? (*Hint:* Consider joining three copies of the polygon, then four copies, then five copies. There may be more than one way to use a particular polygon to create a polyhedron. It also could happen that the sum of the angle measures is less than 360° and the formation doesn't give a polyhedron.)

Regular Polygon	Angle Measure
Equilateral triangle	
Square	
Pentagon	
Hexagon	
Heptagon	

Table 10 Angle Measures of Regular Polygons

2. Fill in the following chart to summarize your findings.

Polygon	Number Meeting at Vertex	Angle Sum (<360°)
Equilateral triangle		
Equilateral triangle		
Equilateral triangle		
Square		
Pentagon		

Table 11 Vertex Angle Sum

3. Why isn't there a row for a hexagon in the chart?

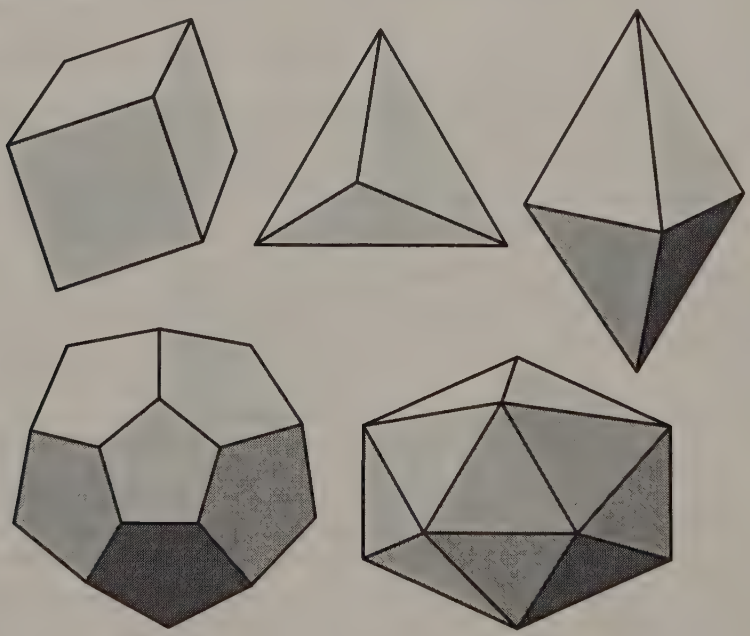
4. Explain why there are no other regular polygons that could be used to construct a regular polyhedron.

There are five regular polyhedra. Three of the regular polyhedra are made from triangles, one is made from squares, and the last is made from pentagons. Each of the regular polyhedra has a name based on the Greek prefixes for a number and the suffix *hedron,* for "seat" or "face."

Prefix	Number of Faces
tetra	4
hexa	6
octa	8
dodeca	12
icosa	20

According to the chart, a **tetrahedron** has four faces, a **hexahedron** has six, an **octahedron** has eight, a **dodecahedron** twelve, and an **icosahedron** twenty. Note that the names tell you the number of faces the polyhedron has, but not which polygon was used to construct the polyhedron.

Study the following diagrams of polyhedra, determine the number of faces for each polyhedron, and write the appropriate name below each. (*Hint:* Visualize carefully. Remember that these are two-dimensional diagrams of three-dimensional objects; part of the object cannot be seen). In the next lesson, you will have an opportunity to make models of each of the Platonic solids, so you will gain a better understanding of them.



The five regular polyhedra have fascinated humanity for centuries. Pythagoras studied the five regular polyhedra around 500 B.C. These polyhedra are often called the Platonic solids, because they were described in detail by Plato around 400 B.C.

The ancient Greeks thought that the world was composed of four elements: fire, water, earth, and air. They used the Platonic solids to represent these elements. The tetrahedron represented fire, the cube (hexahedron) represented earth, the octahedron represented air, and the icosahedron represented water. The twelve faces of the dodecahedron corresponded to the twelve signs of the zodiac.

Later Johannes Kepler, a German astronomer, wondered if there was a relationship between the five Platonic solids and the six planets (only six planets were known at the time: Mercury, Venus, Earth, Mars, Jupiter, and Saturn). He hypothesized that the planets moved on imaginary spheres nested within a set of the five polyhedra.

WRAP-UP

A regular polyhedron (or Platonic solid) is a polyhedron constructed from copies of a single type of regular polygon. Describe the features of a polyhedron.

HOMEWORK

1. Summarize what you have learned in this lesson by completing the following table.

Name	Number of Faces	Polygon Used	Angle Sum of Vertex
Tetrahedron			
Hexahedron (cube)			
Octahedron			
Dodecahedron			
Icosahedron			

Table 12 Polyhedra Summary

2. Find and describe at least three examples of natural or human-made objects that are shaped like regular polyhedra.

3. Explain why a Platonic solid cannot be constructed from hexagons.

4. An angle sum of less than 360° would yield a polyhedron; an angle sum of exactly 360° can yield a tessellation. Can an object be constructed using polygons and vertices with an angle sum greater than 360°? If so, describe the object(s). If not, explain why not.

5. There are many polyhedra that are not regular polyhedra. Some are constructed from several different regular polygons; others are constructed from irregular polygons. *Archimedean solids* are not regular polyhedra. Go to the library or search the World Wide Web to answer the following questions.
   a. Who was Archimedes?

   b. What is an Archimedean solid?

   c. How do Archimedean solids differ from Platonic solids?

   d. How many Archimedean solids are there?

e. What human-made or naturally occurring objects are in the shape of Archimedean solids?

f. Provide illustrations of the more interesting Archimedean solids.

EXTENSIONS

Go to the library or search the World Wide Web to gather information on the following topics.

1. As noted earlier, Johannes Kepler had an interesting hypothesis about the relationship between the Platonic solids and the six known planets of his time. Research this relationship and prepare a paper detailing the nesting of the spheres of the planets and the Platonic solids. Including an illustration of the hypothetical system would be helpful.

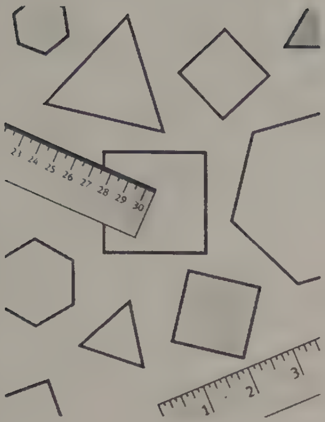
2. Many polyhedra are very attractive. Find a diagram or model of each of the following:

   Rhombicosadodecahedron

   Truncated cubeoctahedron

   Truncated icosahedron

3. There are a number of sites on the World Wide Web that provide information about polyhedra. Use a search engine and look for sites using the keywords *polyhedron* or *geometry*.



LESSON 6: Building Polyhedra

Introduction

In this lesson you will build polyhedra from regular polygons. One of the best ways to really understand polyhedra is to build a set of models. Models can be made from paper, cardboard, coffee stirrers, popsicle sticks, or Tinker-Toys. Even forks and vegetables have been used. In this lesson you will find instructions for constructing paper models of the five Platonic solids.

Learning Objectives
In this lesson, you will . . .
- create models of polyhedra

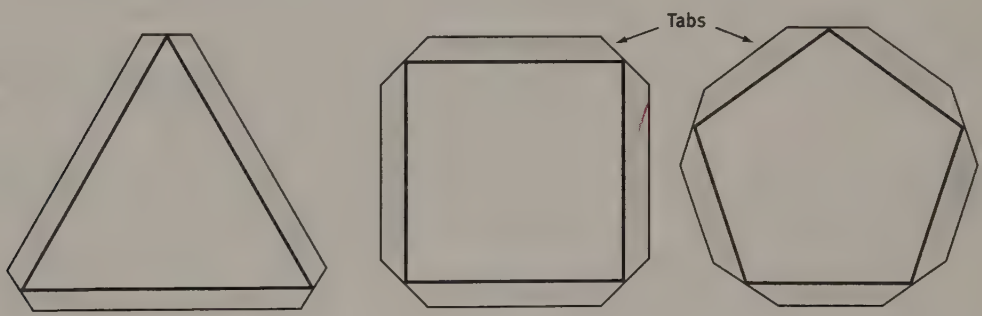
Materials
- protractor, ruler
- high-quality art paper, glue
- scissors

Building Polyhedra

The first step in making a regular polyhedron is to make a pattern for the regular polygon that forms its sides. Recall that three of the regular polyhedra are constructed from equilateral triangles (tetrahedron, octahedron, and icosahedron), one from squares (hexahedron or cube), and one from pentagons (dodecahedron). Accordingly, you only need three patterns: one each for the triangle, square, and pentagon.

Follow the instructions to create patterns for the triangle, square, and pentagon using a protractor and ruler. There may be easier ways to draw triangles, squares, and pentagons, but models require very accurately drawn polygons.

Draw the patterns on very stiff paper or light cardboard. Cereal box cardboard or a file folder works well. Measure carefully and accurately. The sides of all the polygons should be 3 inches long. Hold the protractor and ruler with a steady hand. Do not cut out your patterns until tabs have been drawn on all sides.



ACTIVITY 1—MAKING PATTERNS

1. *Equilateral Triangle.* Recall that the measure of an angle in an equilateral triangle is 60°.
   - Use the ruler and a pencil to draw a line exactly 3 inches long.
   - Place the center of the protractor at one end of the line you just drew and make a mark to indicate a 60° angle.
   - Use the ruler and pencil to join the end of the line and the mark indicating the 60° angle. Make sure the joining line is exactly 3 inches long.
   - Use the ruler to complete the triangle.
   - Verify that all sides are 3 inches long and that all angles are 60°.
   - Draw quarter-inch tabs on all three sides of your triangle.
   - Carefully cut out your pattern.

2. *Square.* Recall that the measure of an angle in a square is 90°.
   - Use the ruler to draw a line exactly 3 inches long.
   - Place the center of the protractor at one end of the line you just drew and make a mark to indicate a 90° angle.
   - Draw a 3-inch line between the end of your original line and the mark made with the protractor. Now you have a 90° angle.
   - Repeat the previous step at the other end of the original line.
   - Use the straightedge to connect the two new lines for the fourth side.
   - Verify that all sides are 3 inches long and that all angles are 90°.
   - Draw quarter-inch tabs on all four sides of your square.
   - Carefully cut out your pattern.

3. *Pentagon.* Recall that the measure of an angle in a regular pentagon is 108°.
   - Use the ruler and a pencil to draw a line exactly 3 inches long.
   - Place the center of the protractor at one end of the line you just drew and make a mark to indicate a 108° angle.
   - Use the ruler and pencil to join the end of the original line and the mark indicating the 108° angle. Make sure the joining line is exactly 3 inches long.
   - Repeat the previous two steps twice by placing the protractor at the end of any line that has not yet been used and then marking the 108° angle.
   - Use the ruler to complete the pentagon.
   - Verify that all sides are 3 inches long and that all angles are 108°.
   - Draw quarter-inch tabs on all five sides of your pentagon.
   - Carefully cut out your pattern.

ACTIVITY 2—CONSTRUCTING REGULAR POLYHEDRA

To construct a model of a regular polyhedron, enough copies of the regular polygon used to form the polyhedron must be cut from art paper or high-quality construction paper. Fill in the following table to ensure that you know the number and type of polygon needed to form each polyhedron.

Name	Type of Polygon	Number of Polygons (faces)
Tetrahedron	triangle	4
Cube		
Octahedron		
Dodecahedron		
Icosahedron		

Table 13 Building Polyhedra

General guidelines for constructing models

- Carefully trace the pattern of the appropriate polygon onto art paper or high-quality construction paper. Typical construction paper does not fold or glue very well.
- Trace and cut out a few more faces than are needed to make the polyhedron. You may make some mistakes in assembly.
- Carefully score and fold along the edges of the tabs.
- Constantly study the pictures of the polyhedron you are trying to construct as you put the pieces together.
- Do not use too much glue. It will be slow to dry and may cause the paper to dissolve.
- Wait until the glue from a recent join is sufficiently dry before attaching another piece.
- You might find it easier to remove some of the tabs as you put the polyhedra together.
- Avoid crushing any of the pieces. Handle them carefully.
- You can use your time efficiently if you work on several models at the same time. While a new join on one model is drying, make a new join in another model, and so on. Be careful to not get confused.
- Working with others will give you the extra hands you will need.

HOMEWORK

1. If you have not done so already, complete all models of the regular polyhedra.

2. Explain why the regular polyhedra are typically listed in the order tetrahedron, cube, octahedron, dodecahedron, and icosahedron.

3. What's another name for a cube? What does it mean?

4. What language do the names of the polyhedra come from?

5. A **net** is a two-dimensional figure that can be folded to make a three-dimensional object. For example, in order to produce a cereal box, a company cuts out one piece of cardboard and then folds it. In other words, they do not glue six rectangles together. What does the net for a cereal box look like? (Cut one open to see.) Sketch it below.

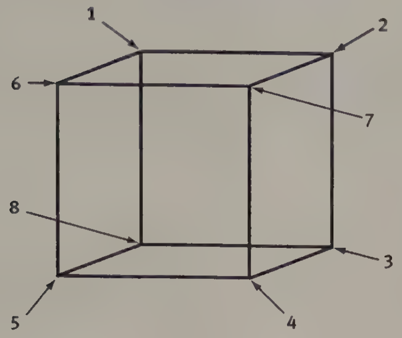
6. What does the net for a tetrahedron look like?

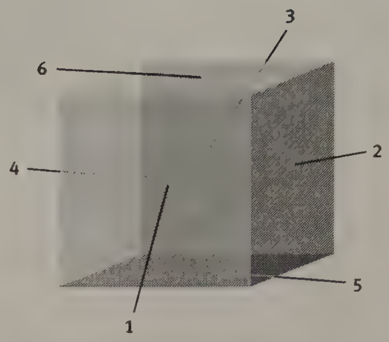
7. Which polyhedra have rotational symmetry? Which have reflection symmetry? Describe the symmetries of each of the polyhedra. Sketches may be helpful.

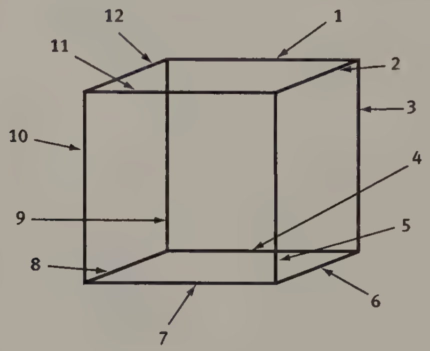
EXTENSION

In this lesson, we used a ruler and a protractor to draw a triangle and a square. In formal Euclidean geometry this is not allowed. The only tools allowed are a compass and an unmarked straightedge. Look in a geometry book or on the World Wide Web for the Euclidean construction of a triangle and of a square. Perform those constructions here.

LESSON 7: Feeling a Bit Edgy







Introduction

Polyhedra are literally the building blocks of many three-dimensional objects you come in contact with every day. Now that you have models of the polyhedra, you are ready to analyze their characteristics further.

Learning Objectives
In this lesson, you will . . .
- examine the relationship among vertices, edges, and faces of regular polyhedra

Materials
- the models you created in Lesson 6

Feeling a Bit Edgy

Polyhedra have many interesting properties. One interesting property involves the relationship between the number of vertices, edges, and faces in a polyhedron. Recall that a vertex is a point at which the corners of the polygons used to form a polyhedron meet. An edge is a line segment along which two polygons meet. Finally, a face is the surface of a polygon.

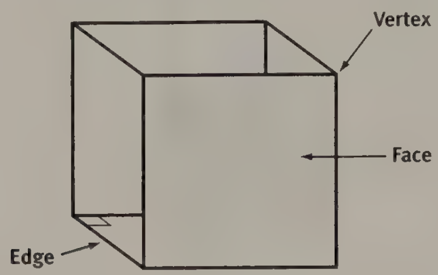


Figure 3 A Hexahedron

Refer to your earlier work to complete the following table.

Name	Vertices	Edges	Faces
Tetrahedron	4	6	4
Cube			
Octahedron			
Dodecahedron			
Icosahedron			

Table 14 Characteristics of the Platonic Solids

Study the numbers in the table. There is a pattern involving the number of vertices, edges, and faces for each polyhedron.

On the facing page you will find a series of questions and steps designed to help you find the pattern and use the pattern to write a formula describing the relationship among the numbers of vertices, edges, and faces. Use a spare piece of paper to cover up the questions, steps, and final answer until you are ready to see them.

Try to determine the pattern yourself. If you get stuck, allow yourself to look at one question or step. After answering a question or performing a step, try again to find the relationship before looking at the next question or step.

Consider each *row* separately. You are looking for a pattern that each polyhedron exhibits (not for a pattern in the columns).

Consider combining (with addition or subtraction) the number of vertices (V), edges (E), and faces (F) for each polyhedron.

Try different combinations of adding and subtracting the vertices, edges, and faces. Do you observe a pattern?

Name	$V + E + F$	$V + E - F$	$V - E + F$	$V - E - F$
Tetrahedron				
Cube				
Octahedron				
Dodecahedron				
Icosahedron				

Table 15 A Relationship Among Vertices, Edges, and Faces

Euler–Descartes Formula for polyhedra

Write the formula here:

As the above name suggests, the relationship among the numbers of vertices, edges, and faces of a polyhedra was discovered by Leonhard Euler and René Descartes. Interestingly, René Descartes stated the fact first about 1635. Without any knowledge of Descartes's work, Leonhard Euler, in 1752, also stated the formula.

This formula holds for any polyhedron (not just regular polyhedra) that does not have a hole. A hole is like a tunnel; it goes completely through the polyhedron.

WRAP-UP

The five regular polyhedra exhibit a relationship among the numbers of vertices, edges, and faces. What is that relationship?

HOMEWORK

1. Name the five regular polyhedra. Be sure to spell each word correctly. Use a mathematics dictionary to find the correct pronunciation of each word.

2. The Euler–Descartes Formula can be stated in many different ways. Solve the formula $V - E + F = 2$ for each of the indicated variables to show the different ways of stating the formula.

$V =$ $V + F =$

$E =$ $V - E =$

$F =$

3. Find three objects at home that are not regular polyhedra, and verify that the Euler–Descartes Formula still holds.

Object	Vertices	Edges	Faces	$V - E + F = 2$

Table 16 Euler–Descartes Formula for Objects at Home

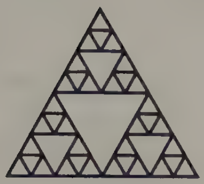
EXTENSIONS

1. The Euler-Descartes Formula holds for all polyhedra that do not have holes. There is another, similar formula for polyhedra that have holes. Find the formula for polyhedra that have holes. Find examples of at least two polyhedra that have holes, and confirm that the new formula holds.

2. Leonhard Euler had a variety of mathematical interests besides polyhedra. Research his contributions to mathematics, and write a brief paper or create a poster describing and illustrating at least two of his contributions.

3. René Descartes had a variety of mathematical interests besides polyhedra. Research his contributions to mathematics, and write a brief paper or create a poster describing and illustrating at least two of his contributions.

LESSON 8: Fractal Patterns



Introduction

We use the words *pattern* and *trend* frequently. We can better describe and understand patterns or trends by using the languages of tessellation, geometry, or algebra. However, there are many items and behaviors that we cannot describe this way. For example, we often look for patterns in clouds, study the beauty of the patterns in snowflakes, worry about the patterns in a loved one's electrocardiogram, and plan vacations on the basis of weather patterns. All of these patterns are difficult if not impossible to describe by using the vocabulary of algebra or basic geometry. For the most part, the patterns just described are very complex. They are sometimes even considered random or chaotic—not patterns at all.

Learning Objectives

In this lesson, you will . . .

- describe the properties of fractals: self-similarity, iteration, and infinite complexity
- construct fractals using an initiator and a generator
- construct fractals by following rules

Fractal Patterns

A relatively new field of mathematics, fractal geometry, is helping to provide a means of describing, viewing, and exploring seemingly complex patterns. Fractal geometry is the study of complex patterns that are generated by performing a simple rule over and over again. We call these complex patterns **fractals.** For example, we call the fractal image at the beginning of this lesson the Sierpinski Gasket and construct it by repeating the following steps over and over again.

1. Draw a triangle.
2. Locate the midpoint of each side of each of the triangles.
3. Draw lines connecting the midpoints to form new inside triangles.
4. Cut out the new inside triangle.

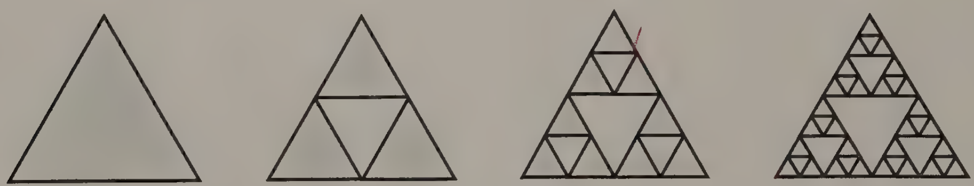


Figure 4 First Four Construction Steps of the Sierpinski Gasket

Note that after the first construction step, the resulting gasket contains three copies of the previous gasket. Similarly, after the second construction step, the resulting gasket contains three copies of its previous gasket, and this continues. We call this property of containing smaller copies of itself **self-similarity.** This is a property of fractal images.

Also note that the set of steps always transforms the previous gasket into the next gasket. We call the process of applying a rule over and over again, and using the previous output as input for the next stage, **iteration.** We construct fractal images through iteration of a variety of rules.

In theory, we can repeat the construction steps infinitely many times; we call the resulting construction the **limit** of the fractal. In other words, we should be able to zoom in on the fractal image and see more and more detail each time, and the detail is similar to the original fractal image. This is because the construction rule has been applied at all possible levels. This **infinite detail** or complexity is a fundamental property of ideal fractal images.

In reality, we can continue the steps only as long as our patience holds out or until our drawing tool is no longer able to apply the rule such that a difference is noticeable. Thus any fractal images that we draw are approximations of the ideal fractal image.

In summary, fractals exhibit some kind of self-similarity, have infinite detail or complexity, and are constructed through iteration of simple rules.

As an alternative to specifying a rule as a sequence of steps, an **initiator** and **generator** can be given. An initiator is a starting point for the construction of the fractal image, and a generator is a pattern that shows the transformation to be performed on the previous construction to create the next construction. Reduced copies of the generator replace all appearances of the initiator in each step of construction.

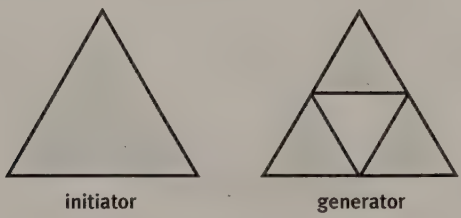


Figure 5 Initiator and Generator for Sierpinski Gasket

We can construct another set of fractal images, known as Pythagorean Trees, by iterating the following steps.
1. Draw a square.
2. Attach an isosceles triangle to one of its sides along the triangle's base.
3. Attach two squares along the free sides of the triangle.

You can also construct a Pythagorean Tree by using the following initiator and generator.

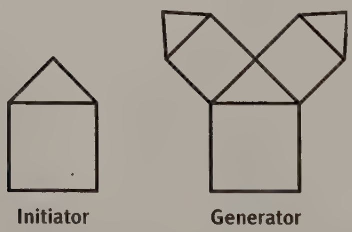


Figure 6 Initiator and Generator for a Pythagorean Tree

In each step of construction of a Pythagorean Tree, we transform all appearances of the initiator in the previous step into copies of the generator.

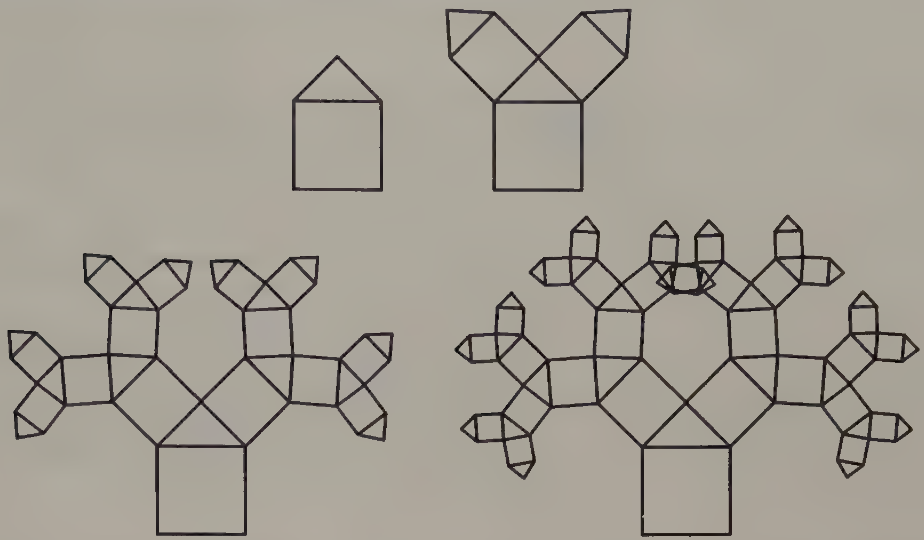


Figure 7 Several Construction Steps of a Pythagorean Tree

The Pythagorean Tree gets "bushier" (more complex) after each iteration of the construction rule. Thus the ideal Pythagorean Tree is infinitely bushy (has infinite detail at its limit). Pythagorean Trees are obviously self-similar, because any complete branch looks like a smaller version of the original tree.

There are infinitely many Pythagorean Trees; the type of tree constructed depends on the shape of the triangle. The tree constructed above is bilaterally symmetrical because we used an isosceles triangle. Scalene triangles produce fern-like fractal images.

Equilateral triangles produce images very much like some of the tessellations studied earlier.

After the early construction stages, the Pythagorean Tree begins to look like some familiar natural objects. It looks like a sea urchin, broccoli, or a tree. A number of fractal images or parts of fractal images resemble natural phenomena (such as leaves, clouds, coastlines, heart rhythms, or galaxies). This property of fractals is what makes them so useful for modeling and studying nature.

✎ Use the following initiator and generator to construct a fractal image. Sketch at least four stages of construction.

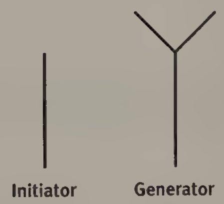


Figure 8

Describe the three common properties of fractals (self-similarity, construction by iteration, and infinite detail) as they are related to the fractal image you have sketched.

ACTIVITY 1—CREATING THE KOCH CURVE

1. Iterate the following steps to construct another fractal image. We call the fractal that you will construct the Koch Curve. Hexagonal dot paper makes the construction a little easier.
   - Draw a line segment.
   - Divide the line into three equal lengths.
   - Draw an equilateral triangle on top of the line, using the middle third of the original segment for the base of the triangle.
   - Erase the base of the triangle just drawn.
   - Repeat, starting with the second step. (You now have four lines.)

   Sketch at least four stages of construction.

2. When you connect three Koch curves, the resulting fractal image is called a Koch Snowflake or Koch Island. Connect your curve with the curves of two classmates to form a Koch Snowflake.

3. Describe the three common properties of fractals (*self-similarity,* construction by *iteration,* and *infinite detail*) as they are related to the fractal image you have sketched.

Many fractal images resemble natural phenomena. Examples of fractal-like images in nature include trees, leaves, snowflakes, electrocardiograms, coastlines, mountain ranges, cauliflower, and the surface of the moon. Cauliflower exhibits self-similarity; a single branch of cauliflower resembles an entire head. A small cloud or a piece of a cloud looks like an entire cloud.

We call these natural phenomena *fractal-like* because they are not ideal fractals. Recall that ideal fractals have infinite detail; no matter how often or how far we zoom in, we should see even more detail, and that detail should resemble in some way the original fractal. If we break off a branch of cauliflower, we get a smaller head of cauliflower, and if we repeat the process, we get an even smaller head of cauliflower. However, we cannot repeat the process infinitely, because eventually we are holding nothing but mush or air. Cauliflower does not have infinite detail. The same is true of clouds, trees, mountain ranges, etc.

Fractal-like objects in nature are not ideal fractals, but they resemble the early stages of fractal construction. Because of the similarity between ideal fractals and fractal-like natural phenomena, ideal fractals can and have been used to model natural phenomena. Scientists have used fractals to model weather patterns, heart rhythms, water and air turbulence, and entire planets. The predictions based on the fractal models have been encouragingly accurate.

WRAP-UP

Complete each of the following.

1. To construct a fractal:

2. Fractals exhibit:

3. There are many fractal-like phenomena in nature, including:

HOMEWORK

1. Explain why mountain ranges are considered to be fractal-like.

2. Look for at least three fractal-like objects in the world (try to find examples that have not been given in this lesson). In particular, look for things that are self-similar. Describe each object, and explain in what way each is self-similar. If the object exhibits the other properties of fractals, describe those as well.

3. There are many sites on the World Wide Web that provide information about fractals. Do a keyword search on *fractal* or on the name of one of the fractals mentioned in this lesson. Write a few sentences communicating the information you found.

EXTENSIONS

Fractals and Drawing Programs

A drawing program that allows drawing of lines, rotation, reflection, scaling, copying, pasting, and grouping of objects can allow construction of some highly detailed fractal images.

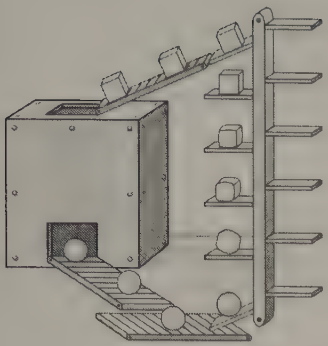
1. For example, the Sierpinski Gasket can be constructed with a drawing program by using the following steps.

   1. Create a filled triangle.
   2. Scale the triangle down by a factor of 3.
   3. Make two more copies of the scaled-down triangle.
   4. Paste the three triangles together like a Sierpinski Gasket (corners of the triangles touch) to create a new triangle.
   5. Repeat from Step 2 until the desired level of detail is reached.

2. A Koch Island can be constructed with a drawing program by using the following steps.

   1. Draw a line.
   2. Make three more copies of the shape.
   3. Rotate one of the copies by 60°, and rotate another by −60°.
   4. Paste the lines together to form a shape like 
   5. Group and scale down the new shape by a factor of 3.
   6. Repeat from Step 2 until the desired level of detail is reached.
   7. Make two more copies of the Koch Curve.
   8. Rotate one of the copies by 120°, and rotate another by −120°.
   9. Paste and group the curves together to form a Koch Island.

LESSON 9: Recursive Relations



Introduction

In the previous lesson, you explored the idea of iteration geometrically. You found that with an initiator and generator, it is possible to repeat a process infinitely many times. There are also symbolic ways to define an iteration process. Using variables and equations, we can define a numerical iteration.

Learning Objectives:
In this lesson, you will . . .
- begin using recursive relations

Recursive Relations

Recall the important properties of the fractals studied in the previous lesson: self-similarity, infinite detail, and construction by iteration. Many of these rules consisted of a sequence of steps indicating how to draw a fractal image by transforming an existing image. Alternatively, an initiator and a generator were shown, and we applied the generator repeatedly to add detail to the fractal image.

This repetition of rules, with one result being transformed to yield the next result, can also be specified by a mathematical equation known as a **recursive relation.** An example of a recursive relation is

$$a_n = 2a_{n-1} + 4$$

We call the small n and $n-1$ below the a's **subscripts**. They indicate the number of repetitions of the rule that we apply to calculate the a under consideration. The relation states that to get the nth (next) result transform the $(n-1)$th (previous) result by multiplying it by 2 and adding 4. In this way, a recursive relation can serve as a generator for a sequence of numbers. However, before a sequence of numbers can be generated, an initiator must be specified. This can be done as follows:

$$a_1 = 3$$

The subscript 1 indicates that 3 is the *first* number in the sequence. To calculate the next number, use the recursive relation $a_n = 2a_{n-1} + 4$. According to the recursive relation, calculate the second number by multiplying the first number by 2 and then adding 4. The first five construction steps of the sequence appear below.

$$a_1 = 3$$

$$\begin{aligned} a_2 &= 2a_1 + 4 \\ &= 2 \cdot \mathbf{3} + 4 \\ &= 6 + 4 \\ &= \mathbf{10} \end{aligned} \qquad \begin{aligned} a_3 &= 2a_2 + 4 \\ &= 2 \cdot \mathbf{10} + 4 \\ &= 20 + 4 \\ &= \mathbf{24} \end{aligned}$$

$$\begin{aligned} a_4 &= 2a_3 + 4 \\ &= 2 \cdot \mathbf{24} + 4 \\ &= 48 + 4 \\ &= \mathbf{52} \end{aligned} \qquad \begin{aligned} a_5 &= 2a_4 + 4 \\ &= 2 \cdot \mathbf{52} + 4 \\ &= 104 + 4 \\ &= \mathbf{108} \end{aligned}$$

Compare the properties of fractals with a recursive relation. First, a sequence of numbers is generated by repeatedly applying a rule that accepts as input the previous result (*iteration*). Second, in theory, the sequence could go on indefinitely; this fact is analogous to the *infinite detail* property of the geometric fractals constructed in the previous lesson. Finally, the sequence does exhibit *self-similarity*; each number in the sequence is an arithmetic transformation of the previous number. In other words, you calculate each number in the sequence by using a similar formula.

Consider the formula $a_n = 7 \cdot 2^{(n-1)} - 4$.

Verify that the formula given above, $a_n = 7 \cdot 2^{(n-1)} - 4$, calculates the same numbers as the recursive relation. Evaluate the formula for a_1, a_2, a_3, a_4, and a_5, and compare the results with the sequence calculated earlier using the recursive relation.

$a_1 = 7 \cdot 2^{(1-1)} - 4 = 7 \cdot 2^0 - 4 = 7 \cdot 1 - 4 = 3$

$a_2 =$

$a_3 =$

$a_4 =$

$a_5 =$

ACTIVITY 1—PRACTICING WITH RECURSIVE RELATIONS

You will work with a few recursive relations and examine the pattern of values in the resulting sequence.

1. For the recursive relation $B_n = 0.5B_{n-1} + 0.5$, $B_1 = 2$, calculate the first ten terms in the sequence.

$B_1 = 2$ $\quad$ $B_2 =$ $\quad$ $B_3 =$ $\quad$ $B_4 =$ $\quad$ $B_5 =$

$B_6 =$ $\quad$ $B_7 =$ $\quad$ $B_8 =$ $\quad$ $B_9 =$ $\quad$ $B_{10} =$

2. If you were to continue the above interative process, is there a single value that the numbers in the list seem to be getting close to? If so, what number is it?

3. For the recursive relation $B_n = 0.5B_{n-1} + 0.5$, $B_1 = 4$, calculate the first ten terms in the sequence.

$B_1 = 4$ $B_2 =$ $B_3 =$ $B_4 =$ $B_5 =$

$B_6 =$ $B_7 =$ $B_8 =$ $B_9 =$ $B_{10} =$

4. If you were to continue the above interative process, is there a single value that the numbers in the list seem to be getting close to? If so, what number is it?

5. Try other values for B_1 in your group. Have each member of the group start with a different value of B_1. Consider starting with negative values. Compare the ultimate behavior you obtained with the results that other groups obtained.

6. What can you say about the effect of the starting value on the ultimate behavior of the sequences you generate?

Recursive relations and limits

Recall that there is a difference between ideal fractals and the fractals that we can draw or calculate in a reasonable amount of time or with a reasonable amount of precision or clarity. In particular, we can construct an ideal fractal by iterating the construction rule indefinitely. One goal of applying a rule an infinite number of times is to determine the **limit** of the fractal or sequence. You can think of the limit of a fractal or sequence as the ideal object toward which all the intermediate constructions tend. Sometimes such a limit exists and sometimes it does not.

Consider the recursive relation $a_n = a_{(n-1)}^2$.

If the initiator is $a_1 = 2$, the sequence of numbers generated is 2, 4, 16, 256, 655536, 4294967296, ... Note that the numbers generated grow larger and larger with each step. In addition, the difference in value between successive numbers gets larger and larger. The numbers do not tend toward any particular number. Therefore, we say this sequence has no limit.

Now consider the same recursive relation with the initiator $a_1 = 0.5$. Now the sequence of numbers generated is 0.5, 0.25, 0.0625, 0.00390625, 0.0000525878906, 0.0000000002328306437, . . . Note that the numbers generated get closer and closer to zero with each step. The numbers get smaller and smaller, but they do not become negative, and the difference in value between succeeding numbers gets smaller with each iteration. In such a case, we say that the limit of the sequence is zero, because zero is the number toward which the numbers in the sequence tend.

Iterate the following recursive relations and determine whether they have limits. Carefully explain, in a few sentences, why the relation has or does not have a limit, and give the limit if it exists.

$$a_n = a_{(n-1)}^3 \qquad a_1 = 0.75$$

$$a_n = 2a_{(n-1)}^2 \qquad a_1 = 2$$

$$a_n = a_{(n-1)}^3 \qquad a_1 = 1$$

$$a_n = 2 \cdot a_{(n-1)} \qquad a_1 = 0.5$$

Recursive relations and initial conditions

As we have said, before you can generate a sequence of numbers using a recursive relation, you must specify an initiator. Similarly, an initiator or initial conditions must first be specified to construct a fractal. The need for an initiator or initial condition for recursive relations and fractals is simple to understand. However, you may not have noticed the incredible effect the choice of an initiator can have on the result. Different initiators (initial conditions) can yield very different results. For example, the recursive relation $a_n = a_{(n-1)}^2$ had no limit when the initiator was $a_1 = 2$, but the same recursive relation had a limit of zero when the initiator was $a_1 = 0.5$.

Iterate the following recursive relations and compare with your previous results.

$$a_n = a_{(n-1)}^3 \qquad a_1 = -2$$

$$a_n = 2 \cdot a_{(n-1)} \qquad a_1 = 0$$

Scientists have given a special name to the fact that the choice of initiator can have such a drastic effect on the outcome of iteration. It is called the **Butterfly Effect.** The expression comes from an observation that weather patterns (which are fractal-like) are so chaotic that the fluttering of a butterfly's wings in the rain forest could result in a tornado in Moscow.

The concept of limit and the effect of slight changes in initial conditions are used to create some highly complex fractals. Scientists have used such fractals to model heart rhythms, weather patterns, lightning strikes, air and water turbulence, and other seemingly unpredictable phenomena.

WRAP-UP

Recall that fractals are constructed through the iteration of construction rules. The iteration of rules can also be specified by a mathematical equation known as a recursive relation. An example of a recursive relation follows.

$$a_n = 2a_{n-1} + 4 \qquad a_1 = 3$$

1. The expression involving a_1 is known as

2. In the "Butterfly Effect":

HOMEWORK

1. Generate the first five numbers in the sequence specified by the following recursive relation and initiator.

$$a_n = a_{n-1} + 2 \qquad a_1 = 2$$

2. Generate the first five numbers in the sequence specified by the following recursive relation and initiator.

$$a_n = 4 \cdot a_{n-1}^2 \qquad a_1 = -1$$

3. Determine whether $a_n = 2 \cdot a_{(n-1)}^2$ has a limit for each of the initiators given.

$a_1 = 1$

$a_1 = 0.4$

$a_1 = 2$

4. Consider the recursive relation

$$C_n = 3 \cdot C_{n-1} \cdot (1 - C_{n-1}) \qquad C_1 = 0.5$$

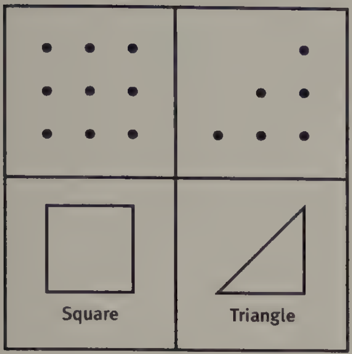
a. Find the first ten terms of this sequence.

$C_1 = 0.5$	$C_2 =$	$C_3 =$	$C_4 =$	$C_5 =$
$C_6 =$	$C_7 =$	$C_8 =$	$C_9 =$	$C_{10} =$

b. If you were to continue the above interative process, is there a single value that the numbers in the list seem to be getting close to?

c. If so, what number is it?

LESSON 10: Square and Triangular Numbers



Introduction

So far, you have worked primarily with geometric patterns. However, number patterns are the focus of most current mathematical investigations. In this lesson, you will examine some number patterns that come from a study of geometric patterns.

Learning Objectives

In this lesson, you will . . .

- improve your ability to recognize visual and numerical patterns
- develop your visual understanding of polygonal numbers
- numerically calculate square and triangular numbers
- derive the formula for the sum of the first *n* counting numbers
- derive the formula for the sum of two consecutive triangular numbers

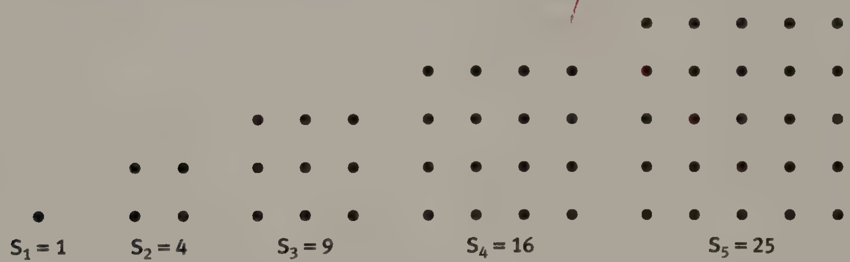
Square and Triangular Numbers

The ancient Greeks worked with **figurate numbers.** We form these numbers by arranging dots (or stones) to form different mathematical figures, such as squares, triangles, pentagons, and other polygons.

The following notation is used in these lessons, so it is important that you understand it and be able to use it as well.

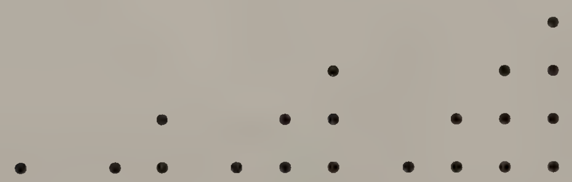
S_1 represents the first square number.	T_1 represents the first triangular number.
S_2 represents the second square number.	T_2 represents the second triangular number.
S_3 represents the third square number.	T_3 represents the third triangular number.
⋮	⋮
S_n represents the *n*th square number.	T_n represents the nth triangular number.

We start our investigation by finding the first seven **square numbers.** We call these numbers *squares* because each can be represented by a square formed by dots.



Continue the pattern and draw S_6 and S_7, the next two square numbers.

If we use the dots to form triangles, we get the **triangular numbers.** The first four triangular numbers are given.



$T_1 = 1$ $T_2 = 3$ $T_3 = 6$ $T_4 = 10$

Continue the pattern and draw T_5, T_6, and T_7, the next three triangular numbers.

Record the first nine square and triangular numbers below. Do you want to draw the eighth and ninth triangular or square numbers? Can you see or guess a pattern instead of drawing the figures?

Square Numbers		Triangular Numbers	
$S_1 =$ ___	$S_6 =$ ___	$T_1 =$ ___	$T_6 =$ ___
$S_2 =$ ___	$S_7 =$ ___	$T_2 =$ ___	$T_7 =$ ___
$S_3 =$ ___	$S_8 =$ ___	$T_3 =$ ___	$T_8 =$ ___
$S_4 =$ ___	$S_9 =$ ___	$T_4 =$ ___	$T_9 =$ ___
$S_5 =$ ___		$T_5 =$ ___	

There is a relationship between consecutive triangular numbers.

Find a formula that describes the relationship between consecutive triangular numbers. Study the triangular numbers and look at the differences between consecutive ones. Do you see any pattern? (Try to find a formula without looking at the hints below. If you need a little help, then feel free to use the hints.)

Hint: Find the values that make each statement true.

$T_2 - T_1 =$ ___ $T_2 = T_1 +$ ___

$T_3 - T_2 =$ ___ $T_3 = T_2 +$ ___

$T_4 - T_3 =$ ___ $T_4 = T_3 +$ ___

When you determine the pattern, write the general relationship:

$T_n - T_{n-1} =$ ______ $T_n = T_{n-1} +$ ______

There is also a relationship between the triangular and the square numbers. First, try the visual investigation. The questions that follow should lead you to an interesting discovery.

Draw a picture or write a formula that shows this relationship.

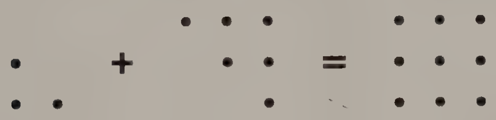
Drawing	Formula
	$T_1 + T_2 = S_2$
	$T_2 + \quad = S_3$
	$T_3 + T_4 = S_4$
	

Table 17 A Relationship Between Square and Triangular Numbers

Can you break a square number into two triangular numbers?

Hint: Consider a square formed by the dots. Try to cut that square into two triangles. Before you answer the questions below, do experiments on different-size squares.

Are the two triangular numbers found the same size? If not, how are they related?

The pictures above strongly suggest that when adding two consecutive triangular numbers (that is, one is one size bigger than the other), we get a square number the same size as the larger triangular number. We will now explore the relationship between triangular and square numbers through an algebraic approach.

You have calculated the first nine triangular numbers and recorded them. Use those values to substitute into the formulas. Look at the two examples that are completed for you. Complete the remaining three in a similar way.

$$T_1 + T_2 = 1 + 3 = 4 = 2^2 \qquad T_2 + T_3 = 3 + 6 = 9 = 3^2$$

$T_3 + T_4 =$ $\qquad$ $T_4 + T_5 =$

$T_5 + T_6 =$

Draw a conclusion and finish the sentences below.

The sum of two consecutive triangular numbers equals ____________.

Use a formula to describe this relationship: $T_{n-1} + T_n =$ ____________.

It is easy to figure out any of the square numbers, $S_1, S_2, S_3, \ldots$, just by calculating the squares of the integers, $1, 2, 3, \ldots$, either in our heads or on a calculator. What about the triangular numbers? So far, when we needed a triangular number, $T_1, T_2, T_3, \ldots$, we had to draw the triangle formed by the appropriate number of dots and count the dots in each triangle. It is easy to do for the small triangular numbers, but what about T_{100}? It would be very useful to have a formula to find any triangular number easily.

We start our next exploration by trying to find the formula that would describe any triangular number.

Verify that the *second* triangular number is the sum of the first *two* positive integers.

Verify that the *third* triangular number is the sum of the first *three* positive integers.

Verify that the *fourth* triangular number is the sum of the first *four* positive integers.

Complete each sentence.

T_{10} is the sum of ____________

T_{100} is the sum of ____________

In general, T_n is ____________

The Double Sum Method

Now we need to find a formula for the sum of the first n positive integers, and then we can apply that formula to calculate any triangular number. We are going to apply the Double Sum Method for finding the sum of the first n positive integers. We will use the letter S to represent the sum of the numbers we are adding.

First, we illustrate the Double Sum Method by an example.

Find the sum, S, of the first four positive integers by using the Double Sum Method.

- Arrange the numbers in increasing order (start with the smallest) and add them.

  $$S = 1 + 2 + 3 + 4$$

- Arrange the numbers in decreasing order (start with the largest) and add them. Because addition is commutative, we must get S again.

  $$S = 4 + 3 + 2 + 1$$

- Add the two equations in columns.
  Adding

  $$S = 1 + 2 + 3 + 4 \quad \text{to}$$

  $$S = 4 + 3 + 2 + 1 \quad \text{yields}$$

  $$2 \cdot S = (1 + 4) + (2 + 3) + (3 + 2) + (4 + 1)$$

- Note that on the right-hand side each sum in the parentheses equals 5 and that we have 4 such sums. Therefore, we get

  $$2 \cdot S = 4 \cdot 5$$

- Divide both sides by 2.

  $$S = \frac{4 \cdot 5}{2}$$

- Simplify the fraction.

  $$S = 10$$

When you are adding just a few numbers, this method does not offer much of an advantage. The advantage comes when you add up the first 100, 1000, or 1 million integers.

- Let S be the sum of the first n counting numbers. Then we write S in two different ways:

  $$\begin{aligned} S &= 1 + \quad 2 \quad + \quad 3 \quad + \cdots + (n-2) + (n-1) + n \\ S &= n + (n-1) + (n-2) + \cdots + \quad 3 \quad + \quad 2 \quad + 1 \end{aligned}$$

- Adding the two equations in columns, we have

  $$\begin{aligned} 2 \cdot S = (1 + n) + (2 + n - 1) + (\qquad\qquad) + \cdots + (\qquad\qquad) + \\ (\qquad\qquad) + (\qquad) \end{aligned}$$

- Note that on the right-hand side, each sum in parentheses equals $n + 1$, so we can simplify inside the parentheses and get

  $$2 \cdot S = (n + 1) + (n + 1) + (\qquad) + \cdots + (\qquad) + (\qquad) + (\qquad)$$

- On the right-hand side, we have n terms of $(n + 1)$ to add, so we can write

  $2 \cdot S =$

- Dividing both sides by 2, we get

  $S =$

Conclusion: To calculate the sum of the first n counting numbers, use the formula

$$S_n = \frac{n \cdot (n + 1)}{2}.$$

We now apply this formula to the problem of calculating any triangular number.

Example: Because T_{100} is really the sum of the first 100 counting numbers, we can easily calculate it by using the formula.

$$\begin{aligned} T_{100} &= \frac{100 \cdot 101}{2} \\ &= 5050 \end{aligned}$$

In general, you calculate any triangular number by using the formula $T_n = \dfrac{n(n + 1)}{2}$, where T_n is the nth triangular number.

Using the formulas for the $(n - 1)$st and nth triangular numbers and adding them, we can also show algebraically that the sum of the $(n - 1)$st and nth triangular numbers is n^2.

$$T_{n-1} + T_n = \frac{(n - 1) \cdot n}{2} + \frac{n \cdot (n + 1)}{2}$$

Adding the fractions, we get

$T_{n-1} + T_n =$

Factoring n out in the numerator, we get

$T_{n-1} + T_n =$

Simplifying the numerator, we get

$T_{n-1} + T_n =$

Simplifying the fraction, we get

$$T_{n-1} + T_n = n^2$$

You have discovered graphically (by drawings) and also algebraically (by calculations) that the sum of two consecutive triangular numbers is equal to a square number $T_{n-1} + T_n = S_n$.

WRAP-UP

1. Triangular numbers are formed by:

2. Square numbers are formed by:

3. The formula ______________ shows the relationship between consecutive triangular numbers.

4. The sum of two consecutive triangular numbers equals a square number. As a formula, this is:

5. The sum of the first n counting numbers can be calculated by the formula

6. Any triangular number can be calculated using the formula

HOMEWORK

1. Calculate T_{10} by drawing the triangular number and counting the dots.

2. Calculate T_{10} by using the formula $T_n = \dfrac{n(n+1)}{2}$.

3. Calculate T_{22} by using the formula $T_n = \dfrac{n(n+1)}{2}$.

4. a. Calculate the square numbers below.

$$1^2 =$$
$$11^2 =$$
$$111^2 =$$
$$1111^2 =$$

b. Predict the value of 11111111^2.

c. Predict the value of 1111111111^2.

d. Compute the value of 11111111111^2 exactly and compare with your prediction.

5. a. Create the three-dimensional version of square numbers by starting with one dot; that is, let $C_1 = 1$. Then create a cube, instead of a square, that has two dots per edge. How many total dots are there in C_2?

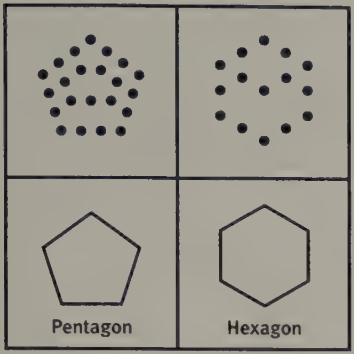
b. Create C_3 by extending the cube for C_2 one dot on each edge and filling in the cube. How many total dots make up C_3?

c. Explain how you would find a formula for C_n. What formula did you find?

6. a. Examine the problem of extending triangular numbers to three dimensions. What sort of geometric shape would you use?

b. Determine the values of the first few of your extended triangular numbers.

LESSON 11: Polygonal Number Patterns



Introduction

After examining triangular and square numbers, it is natural to consider pentagonal and other polygonal numbers. In this section, you will work with pentagonal and hexagonal numbers and develop a new method of predicting the next polygonal number.

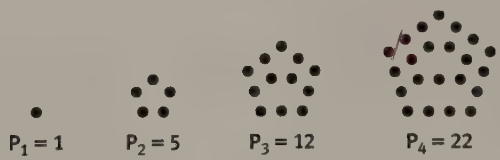
Learning Objectives
In this lesson, you will . . .
- improve your ability to recognize visual and numerical patterns
- calculate the value of polygonal numbers
- examine the relationship between triangular and hexagonal numbers
- use a formula to calculate hexagonal numbers
- explore difference tables for polygonal numbers

Polygonal Number Patterns

We have worked with triangular numbers and square numbers so far. There are many other fascinating polygonal numbers, such as the pentagonal numbers, hexagonal numbers, heptagonal numbers, octagonal numbers, and so on.

Pentagonal numbers

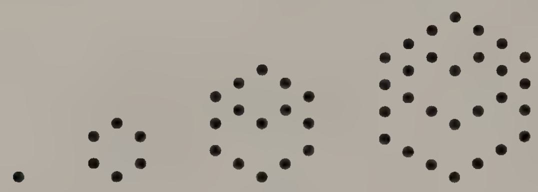
You form **pentagonal numbers** by arranging dots in a pentagon formation. You form the first pentagonal number with one dot. You create the second by forming a pentagon with two dots on each side of the pentagon. You create the third by placing three dots on each side of the pentagon, and so on. Note that each pentagonal number is an extension of the previous one created by adding one dot on each of the top two sides and then finishing the pentagon by surrounding the previous pattern with a five-sided object. Note that each side has the same number of dots.



Draw and determine the value of the fifth pentagonal number.

Hexagonal numbers

You form **hexagonal numbers** by arranging dots in a hexagon formation. Start with one dot. You create the second hexagonal number by forming a hexagon with two dots on each side of the hexagon. You create the third by placing three dots on each side of the hexagon, and so forth. Note that each hexagonal number is an extension of the previous one created by adding one dot each on two of the top sides and then finishing the hexagon by surrounding the previous pattern with a six-sided object. Note that each side has the same number of dots.



$H_1 = 1$ $H_2 = 6$ $H_3 = 15$ $H_4 = 28$

Draw and determine the value of the fifth hexagonal number.

The formula for calculating each of the hexagonal numbers is $H_n = n \cdot (2n - 1)$.

Use this formula to calculate the first four hexagonal numbers.

$H_1 =$

$H_2 =$

$H_3 =$

$H_4 =$

Compare the results of your calculation with the pictures of the hexagonal numbers on the previous page. Is the formula giving the same answer as counting the dots?

Look at the results. Do the numbers look familiar? Where have you seen these numbers before?

Suppose that we look at the case with $n = 3$, $H_3 = 15$. Which triangular number is also 15?

Now consider the case with $n = 4$, $H_4 = 28$. Which triangular number is also 28?

If $n = 3$, then what is $2 \cdot n - 1$?

If $n = 4$, then what is $2 \cdot n - 1$?

Write out the formula for the hexagonal numbers given earlier.

$H_n =$

Recall and write out the formula for triangular numbers.

$T_n =$

In the expression for T_n, replace each n with $2n - 1$. This creates an expression that gives the triangular numbers for the special cases where the index has the form $2n - 1$.

$T_{2n-1} =$

Simplify this formula using the rules of algebra, and compare your result with the formula for H_n.

In general, what is special about $2n - 1$ for every value of n?

Difference tables

We are going to look at the *difference pattern* between consecutive polygonal numbers and construct difference tables for triangular numbers, square numbers, pentagonal numbers, and hexagonal numbers. Once we have the tables, you should look for some number patterns.

You make the tables by listing the polygonal numbers in the first row and then writing the difference between the consecutive polygonal numbers (we call this the *first difference*) in the second row. Calculate the difference between the first difference numbers (we call this the *second difference*), and list these numbers in the third row.

Triangular Numbers	1		3		6		10		15		21		28		36		45
First Differences		2		3		4		5		6		7		8		9	
Second Differences			1		1		1		1		1		1		1		

Table 18 Difference Table for Triangular Numbers

Square Numbers	1		4		9		16		25		36		49		64		81
First Differences		3		5		7		9		11		13		15		17	
Second Differences			2		2		2		2		2		2		2		

Table 19 Difference Table for Square Numbers

✎ How would you work backwards from the second differences to get the next triangular number or the next square number?

ACTIVITY 1—DIFFERENCE TABLES FOR POLYGONAL NUMBERS

1. Continue the pattern to finish the difference table for pentagonal numbers.

Pentagonal Numbers	1		5		12		22		35		51		
First Differences													
Second Differences													

Table 20 Difference Table for Pentagonal Numbers

2. Continue the pattern to finish the difference table for hexagonal numbers.

Hexagonal Numbers	1		6		15		28		45		66		
First Differences													
Second Differences													

Table 21 Difference Table for Hexagonal Numbers

3. Predict the next pentagonal number and the next hexagonal number by working backwards from the second differences.

4. Summarize and write down your observation about the number patterns in the first and second difference sequence.

5. What is the relationship between the second difference and the number of sides, n, of the polygon?

WRAP-UP

1. Pentagonal numbers are formed by:

2. Hexagonal numbers are formed by:

3. Every hexagonal number is also:

4. There is a number pattern in the difference tables of the polygonal numbers. What is it?

HOMEWORK

1. The formula for pentagonal numbers is $P_n = \frac{3n^2 - n}{2}$.

   a. Use the formula to calculate the first three pentagonal numbers.

   b. Compare the results with the pictures of the pentagonal numbers.

2. Every hexagonal number is a triangular number. Is every triangular number a hexagonal number?

3. Make a difference table for the heptagonal numbers. Because you do not know the heptagonal numbers yet and it is very time-consuming to draw them, try to form the difference table backwards. (Start with the second differences, and then figure out the first differences and then the heptagonal numbers.)

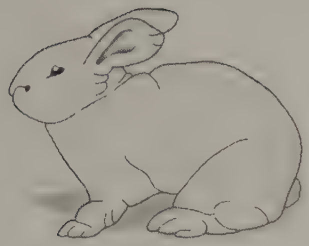
Heptagonal Numbers	1		7							
First Differences										
Second Differences										

Table 22 Difference Table for Heptagonal Numbers

4. a. Examine the problem of extending pentagonal or hexagonal numbers to three dimensions. What sort of geometric shape would you use?

   b. Determine the values of the first few of your extended pentagonal or hexagonal numbers.

LESSON 12: The Fibonacci Sequence



Introduction

In one of the first European books about algebra, the author, who was known as Fibonacci, included a problem about how rabbits reproduce. The solution to this problem led to many fruitful applications of mathematics. In this section, you will examine Fibonacci's problem and a similar problem related to bee families.

Learning Objectives
In this lesson, you will . . .
- explore the Fibonacci Numbers

The Fibonacci Sequence

Leonardo was one of the outstanding medieval Western mathematicians. He was named Leonardo of Pisa after his birth town of Pisa, in Italy. For writing purposes, he nicknamed himself Fibonacci. He is well known as Fibonacci, which means "the son of Bonacci."

In 1202 Fibonacci published the book *Liber Abaci,* in which he introduced Arabic numerals, 0, 1, 2, . . . , to Western culture. This book also posed the following problem, which has inspired people ever since.

The rabbit problem

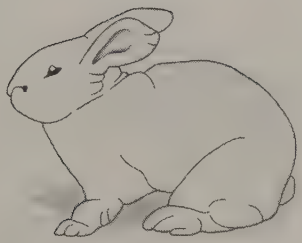
One pair of rabbits (male and female) exist at time zero. After 1 month that pair is still young (not mature enough to reproduce). A month later, at the age of 2 months, they have a pair of babies (one male and one female), and they continue to do so each following month (every month, a new pair is born to the original pair). We will assume that the rabbits live forever.

If each new pair of rabbits starts to reproduce in the same way as described above (giving birth to one male and one female offspring at age 2 months and every month thereafter), how many pairs of rabbits will there be at the end of the first year?

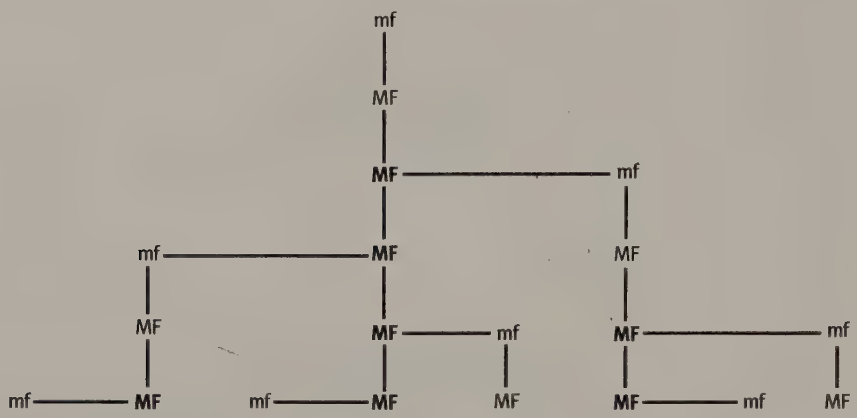
To solve this problem, we will first take a visual approach and then look for a numerical pattern.

Continue the family tree of the rabbit population, representing the population up to the age of seven months. (Add two more generations.)

Use the following notation:



m = newborn male	f = newborn female	(newborn)
M = young male	F = young female	(1 month old)
M = mature male	**F** = mature female	(2 months or older)



It would be difficult to draw the family tree much bigger than the first eight generations. Instead, fill in the table by counting the pairs of rabbits through the first 7 months. Then look back at the numbers and see if you can detect a pattern. You may need to get some help from a classmate. Complete the table by extending the pattern.

Month, *n*	Pairs of Rabbits at the Age of *n* Months	Month, *n*	Pairs of Rabbits at the Age of *n* Months
birth	1	6th	13
1st	1	7th	21
2nd	2	8th	34
3rd	3	9th	55
4th	5	10th	89
5th	8	11th	144

Table 23 Number of Pairs of Rabbits

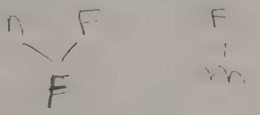
The bee problem

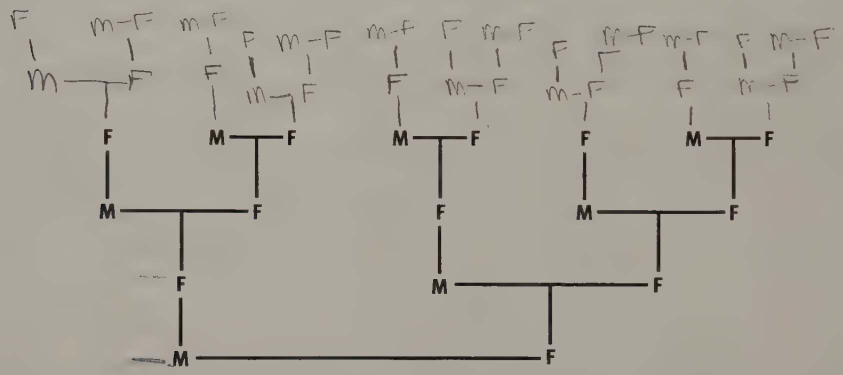
Did you know that a male bee develops from an unfertilized egg (that is, it has a mother but not a father), whereas a female bee develops from a fertilized egg (that is, it has a mother and a father)?

We will do an investigation similar to the one we did with the rabbits. With rabbits, we looked at *descendants*. With the bees, we will look at *ancestors*. We start with a pair of bees and draw a family tree showing the *ancestors*. We start at the root of the tree, which represents the present time, with one male and one female bee. Then we work backwards to their ancestors, and as we move up on the tree, we are considering each previous generation (ancestor) on each level.

Use the following notation: M = male bee, F = female bee.

Continue the construction to show six generations.





Count the number of male bees in each generation (in each level of the tree), and record that number in the table below.

*n*th Previous Generation	Number of Male Bees in the *n*th Generation	*n*th Previous Generation	Number of Male Bees in the *n*th Generation
Current	1	6th	13
1st	1	7th	21
2nd	2	8th	34
3rd	3	9th	55
4th	5	10th	89
5th	8	11th	144

Table 24 Number of Male Bees

Compare the results of the rabbit problem and the bee problem.

The same

The sequence of numbers you found in each of these problems is called the Fibonacci Sequence. In words, describe how this pattern is formed.

$F_n = F_{n-1} + F_{n-2}$

There is a mathematical way to write down the relationship between these numbers representing the pattern that you have found.

Let F_n mean the nth number of the sequence.

$$\left.\begin{aligned} F_1 &= 1 \\ F_2 &= 1 \\ F_n &= F_{n-2} + F_{n-1} \end{aligned}\right\} \quad \textit{Note that this recursive relation has two initiators.}$$

Using the pattern from the recursive relation, we get the following numbers. Complete the equations.

$F_1 = 1$

$F_2 = 1$

$$\begin{aligned} F_3 &= F_1 + F_2 \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} F_4 &= F_2 + F_3 \\ &= 1 + 2 \\ &= 3 \end{aligned}$$

$$\begin{aligned} F_5 &= F_3 + F_4 \\ &= 2 + 3 \\ &= 5 \end{aligned}$$

$$\begin{aligned} F_6 &= F_4 + F_5 \\ &= 3 + 5 \\ &= 8 \end{aligned}$$

$$\begin{aligned} F_7 &= F_5 + F_6 \\ &= 5 + 8 \\ &= 13 \end{aligned}$$

$$\begin{aligned} F_8 &= F_6 + F_7 \\ &= 8 + 13 \\ &= 21 \end{aligned}$$

WRAP-UP

1. The Fibonacci Numbers are constructed by:

2. Fibonacci Numbers can be found by looking at:

HOMEWORK

1. Now that you know the method for creating Fibonacci Numbers, you can answer Fibonacci's question about finding the number of rabbits at the end of the first year. Determine F_{12}.

   $F_{11} + F_{10} =$
   $89 + 144 = 233$

2. Consider the family tree for bees that you explored in this lesson. You found Fibonacci Numbers by looking at the number of male bees in each generation. Can you find Fibonacci Numbers by looking at the female bees at an appropriate part of the tree?

3. Explain how the rabbit tree and the bee tree exhibit the fractal properties of self-similarity, iteration, and infinite detail.

4. Determine first, second and third differences for the Fibonacci Numbers. Explain what you discover.

5. Many natural phenomena are related to Fibonacci Numbers. For example, the number of spirals on a pine cone is usually a Fibonacci Number. The number of petals on a flower is also usually a Fibonacci Number. Look around your house or workplace and try to find more examples of Fibonacci numbers. Try looking up Fibonacci in the library or on the Internet for additional examples.

LESSON 13: The Golden Ratio and the Fibonacci Numbers

Introduction

The Fibonacci Numbers have a relationship to one of the great numbers of antiquity, the Golden Ratio. The Fibonacci Numbers and the Golden Ratio are special because they appear naturally in our world in many places. As a result, many artists and musicians have been attracted to the Fibonacci Sequence and the Golden Ratio and have used them in their art. In this lesson, we will learn about the Golden Ratio and how Fibonacci Numbers can be used to approximate it.

Learning Objectives

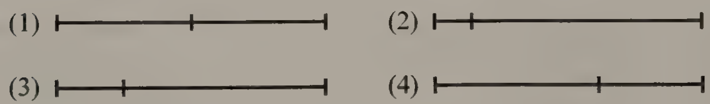
In this lesson, you will . . .

- explore the Golden Ratio
- determine the relationship between Fibonacci Numbers and the Golden Ratio
- apply the Golden Ratio to Golden Triangles and the Golden Spiral

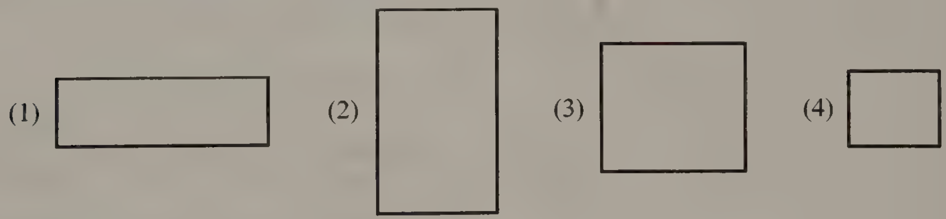
The Golden Ratio and the Fibonacci Numbers

ACTIVITY 1—THE GOLDEN RATIO

1. Look at the pairs of line segments below and circle the pair that is the most pleasing to your eyes.



2. Look at the rectangles below and circle the one that is the most pleasing to your eyes.



3. Record the choices for line segment of each class member in the following table.

	(1)	(2)	(3)	(4)
Number of Students Who Preferred:				

Table 25 Line Segment Preferences

4. Record the choices for rectangles of each class member in the following table.

	(1)	(2)	(3)	(4)
Number of Students Who Preferred:				

Table 26 Rectangle Segment Preferences

According to the ancient Greeks, we were supposed to find the fourth line segment and the second rectangle the most pleasing. Why?

Mathematicians and scientists have been studying proportions for over 4000 years. About 100 years ago, Gustav Fechner tested hundreds of people to find out about their preference of proportion. He concluded that a large percentage of the people found a certain proportion very pleasing. That proportion is the same as the one that people used in Egypt about 4600 years ago in building pyramids. Then, 2300 years ago, the Greeks named it the Divine Proportion or the Golden Ratio.

Let s represent the smaller part (in Question 1 the smaller line segment, in Question 2 the smaller side of the rectangle), and let l represent the longer part (in Question 1 the longer line segment, in Question 2 the longer side of the rectangle).

If s and l satisfy the relation that $\frac{l}{s} = \frac{s+l}{l}$, then this common ratio is called the **Golden Ratio.** One way this can be read is "The ratio of the large to the small equals the ratio of the total to the large."

5. Measure the length of the longer and smaller portions of the line segments in Problem 1. Use those values to complete the table.

Segment	l **(mm)**	s **(mm)**	$\frac{l}{s}$	$\frac{s+l}{l}$
First				
Second				
Third				
Fourth				

Table 27 Line Segment Ratios

6. Measure the length of the sides of the rectangles in question 2. Use those values to complete the table.

Rectangle	l (mm)	s (mm)	$\frac{l}{s}$	$\frac{s+l}{l}$
First				
Second				
Third				
Fourth				

Table 28 Rectangle Ratios

It is most likely that you did not get an exact match in the last two columns of either table. This is due to variations in printing of this book and in the measurement device used. However, in each table, one pair of numbers should be close. The ratios of the fourth line segment and second rectangle should be very close. That is because they were constructed using the Golden Ratio.

The actual value of the Golden Ratio is approximately 1.618033.

A **Golden Triangle** is an isosceles triangle where the long sides are in Divine Proportion to the shorter side. Note that the length of the sides are also Fibonacci numbers.

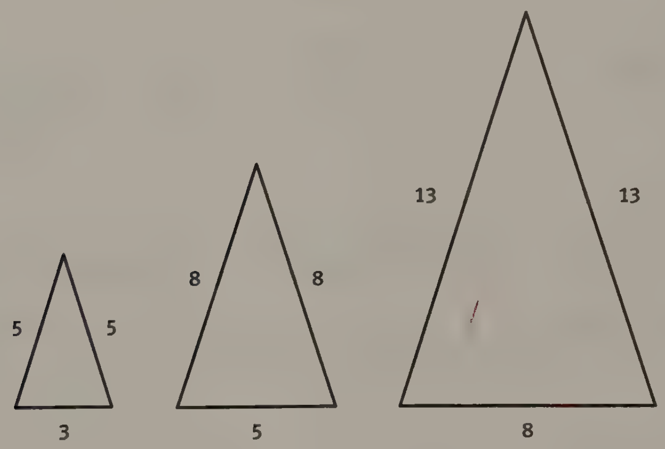


Figure 9 Golden Triangles

The following spiral is called the *Golden Spiral.*

Analyze the picture below. Look for Golden Triangles, for the Golden Ratio, and for the construction pattern. Start with the smallest triangle in the center.

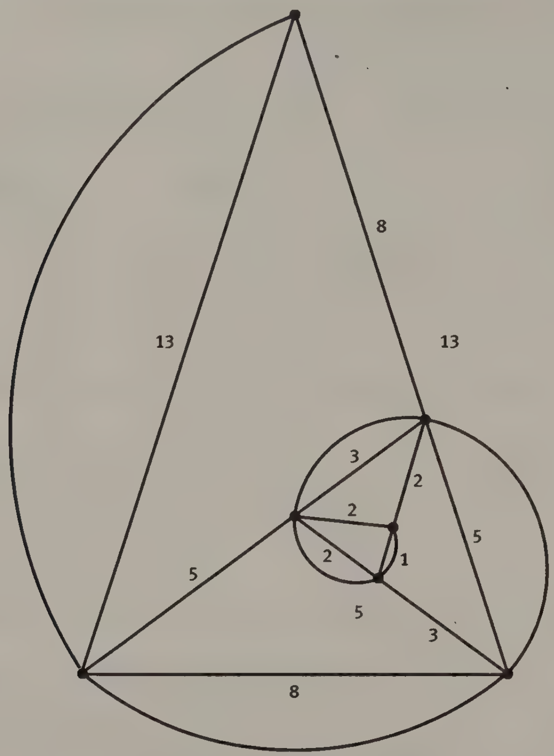


Figure 10 A Golden Spiral

There is a relationship between the Fibonacci Numbers and the Divine Proportion.

Recall the first ten Fibonacci Numbers:

$F_1 =$	$F_2 =$	$F_3 =$	$F_4 =$	$F_5 =$
$F_6 =$	$F_7 =$	$F_8 =$	$F_9 =$	$F_{10} =$

Calculate the ratios of consecutive Fibonacci Numbers and record them in Table 5.29.

Express the ratios with at least six decimal places.

Note: When you are calculating the ratios of succeeding generations, you are actually looking at how the population progresses from generation to generation.

Ratio of $\frac{F_{n+1}}{F_n}$	In Decimals	Ratio of $\frac{F_{n+1}}{F_n}$	In Decimals
$\frac{F_2}{F_1} = \frac{1}{1}$	1	$\frac{F_7}{F_6} =$	
$\frac{F_3}{F_2} = \frac{2}{1}$	2	$\frac{F_8}{F_7} =$	
$\frac{F_4}{F_3} = \frac{3}{2}$	1.5	$\frac{F_9}{F_8} =$	
$\frac{F_5}{F_4} = \frac{5}{3}$	1.666666	$\frac{F_{10}}{F_9} =$	
$\frac{F_6}{F_5} =$			

Table 29 Fibonacci Ratios

The ratios are rapidly approaching a particular number, which is 1.618033 . . . , the famous Golden Ratio.

WRAP-UP

1. If $\frac{l}{s} = \frac{s+1}{l}$ is true for s and l, then the ratio $\frac{l}{s} = 1.618033\ldots$ is called:

2. There is a strong relationship between the Fibonacci Numbers and the Golden Ratio. Describe it.

HOMEWORK

1. Construct your own Golden Spiral on construction paper.

2. Take a ruler and measure several rectangles at your home or office. Determine whether the rectangle exhibits the Golden Ratio.

EXTENSION

1. There is a journal called *Fibonacci Quarterly.* Look it up in a library and find an interesting article in it. Share what you found with your class.

LESSON 14: Pascal's Triangle and Algebraic Patterns

Introduction

Another great European mathematician, Blaise Pascal, developed a number pattern that is also used extensively in algebra. His pattern is referred to as Pascal's Triangle and can be specified by a recursive relation. You can use this triangle of numbers to quickly expand expressions of the form $(x + 1)^n$. In this lesson, you will create Pascal's Triangle and explore its use in algebra.

Learning Objectives:
In this lesson, you will . . .

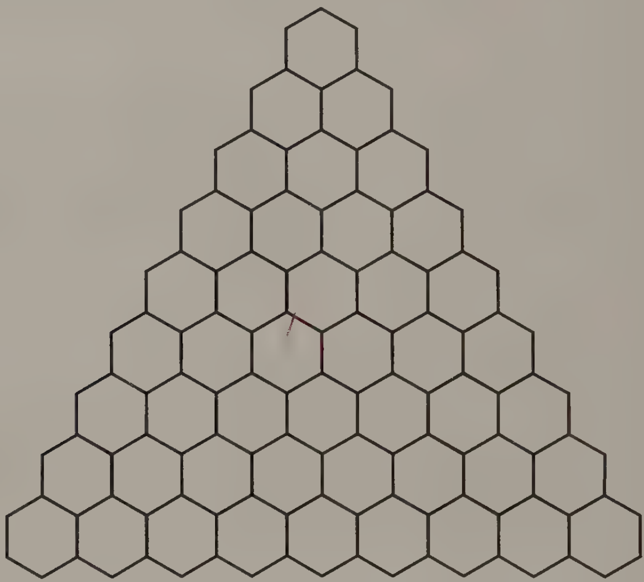
- create Pascal's Triangle
- develop a recursive relation that specifies Pascal's Triangle
- use Pascal's Triangle to expand expressions of the form $(x + 1)^n$

Pascal's Triangle and Algebraic Patterns

ACIVITY 1—CREATING PASCAL'S TRIANGLE

The steps below describe how you construct Pascal's Triangle in the hexagons at the right.

1. To create the top row, write a 1 in the center. We are going to call this the zero-th row.



2. To create the next row, write two 1's diagonally beneath on both sides of the first 1 (one on the left end and one on the right end of the row). This is officially the first row.

3. To create the next row, write a 1 at the left and a 1 at the right end of the row. These should be outside the row above. The middle element is the sum of the two numbers diagonally above it. This is called the second row.

4. To create the next row, write a 1 at the left end and a 1 at the right end of the row, as before. The elements between the 1's (the middle two elements) are the sums of the two numbers diagonally above them. Note that all rows begin and end with 1's.

5. Demonstrate your understanding by completing the next rows of Pascal's Triangle. Continue the construction by following the pattern.

You can use Pascal's Triangle to quickly expand expressions of the form $(x + 1)^n$. Before you can appreciate Pascal's Triangle, you must first recall the standard means for expanding expressions of the form $(x + 1)^n$. Recall that $(x + 1)^2$ means $(x + 1)(x + 1)$, and to expand this expression you must multiply all terms in the first binomial by all terms in the second binomial. We typically use the FOIL (First-Outer-Inner-Last) pattern to help us multiply correctly.

Example:

$$\begin{aligned}(x + 1)(x + 1) &= x^2 + x + x + 1 \\ &= x^2 + 2x + 1\end{aligned}$$

Practice the FOIL pattern on the following multiplication problems:

$(x + 2)(x + 5)$ $\quad$ $(x + 1)(x + 7)$ $\quad$ $(x - 3)(x + 4)$

$(x + 1)^3$ means $(x + 1)(x + 1)(x + 1)$, and to expand this expression you must first multiply $(x + 1)(x + 1)$ and then multiply the result by $(x + 1)$.

Determine $(x + 1)^3$.

Similarly, $(x + 1)^4$ means $(x + 1)(x + 1)(x + 1)(x + 1)$, and to expand this expression you must first multiply $(x + 1)(x + 1)$ and then multiply the result by itself.

Determine $(x + 1)^4$.

Expand each of the following. Be very careful. *Hint:* Remember that anything raised to the zero power is equal to 1, and anything raised to the first power is equal to itself.

$(x + 1)^0 =$

$(x + 1)^1 =$

$(x + 1)^2 =$

$(x + 1)^3 =$

$(x + 1)^4 =$

There is a lot of multiplying to be done, it is very time-consuming, and errors are likely. The correct answers are given in the back. How well did you do?

It would be nice to be able to expand expressions of the form $(x + 1)^n$ without doing so much work and risking so many mistakes. Perform the following tasks to find a relationship between the expansions you just calculated and Pascal's Triangle. The relationship will suggest an easier way to expand expressions of the form $(x + 1)^n$.

Write the coefficients of the expansions given above (that is, erase all the x^n parts and the + symbols). Remember that coefficients are the numbers at the front of a term, e.g. 6 is the coefficient of $6x^2$).

Compare the triangle of coefficients you have remaining with Pascal's Triangle. Explain what you see.

Note that the coefficients of the expansion of $(x + 1)^2$ are 1, 2, 1 and the second row of Pascal's Triangle is 1 2 1. Similarly, the coefficients of the expansion of $(x + 1)^3$ are 1, 3, 3, 1 and the third row of Pascal's Triangle is 1 3 3 1. This suggests that you can find the coefficients of $(x + 1)^5$ in the fifth row of Pascal's Triangle. Remember that the top row is the zero-th row.

List the coefficients of the expansion of

$(x + 1)^5$

$(x + 1)^6$

$(x + 1)^7$

Thus, when we expand, we get

$(x + 1)^5 =$

$(x + 1)^6 =$

$(x + 1)^7 =$

A recursive relation that specifies Pascal's Triangle

Now that you know what Pascal's Triangle is, how to construct it, and how to use it, it is time to consider a recursive relation that specifies Pascal's Triangle. Recall that a recursive relation is a mathematical expression that transforms a previous result to yield the next result. An example of a recursive relation is $a_n = 2a_{n-1} + 4$.

A number in Pascal's Triangle is formed by adding together two numbers from the previous row. Thus the desired form of the recursive relation is (new Pascal's Triangle number) = (number above and to the left) + (number above and to the right). Next we need to consider the subscripts that should be used.

Study the two triangles below. On the left is a small part of Pascal's Triangle. On the right is a triangle of subscripts for the entries in Pascal's Triangle.

1	0,0
1 1	1,0 1,1
1 2 1	2,0 2,1 2,2
1 3 3 1	3,0 3,1 3,2 3,3

According to the preceding triangles, the element in the top row of Pascal's Triangle has the subscript 0,0 indicating the zero-th element of the zero-th row. It may seem unusual to start numbering with "0," but some useful mathematical patterns result. The leading element in the next row should have the subscript 1,0, the next element in that row should have the subscript the 1,1, and the "second" element in the "third" row should have subscript 3,2. Note that the subscript is made up of two parts, a row number and an element number. The row number appears first, then the element number.

Given the above subscript plan, the recursive relation that specifies Pascal's Triangle can be written.

- Let $P_{n,m}$ represent the number being calculated. Then the numbers above and to the left and right, respectively, must be $P_{n-1,m-1}$ and $P_{n-1,m}$.

The recursive relation is written

$$P_{n,m} = P_{n-1,m-1} + P_{n-1,m}$$

Recall, that a recursive relation needs initial conditions. The top element in Pascal's Triangle is 1, so $P_{0,0} = 1$ is an initial condition. In addition, the outer edges of Pascal's Triangle are always 1, so $P_{n,0} = 1$ and $P_{n,n} = 1$, for all n, are also initial conditions. The recursive relation for Pascal's Triangle can now be fully expressed.

Recursive relation for Pascal's Triangle

$$P_{0,0} = 1,\ P_{n,0} = 1, \text{ and } P_{n,n} = 1$$

$$P_{n,m} = P_{n-1,m-1} + P_{n-1,m}$$

for all n and m.

Thus

$$\begin{aligned} P_{4,3} &= P_{4-1,3-1} + P_{4-1,3} \\ &= P_{3,2} + P_{3,3} \\ &= 3 + 3 \\ &= 6 \end{aligned}$$

Use the recursive relation to determine $P_{5,3}$ and $P_{7,2}$.

WRAP-UP

1. Describe the construction of Pascal's Triangle:

2. Pascal's Triangle is a useful mathematical construction that can be specified by a recursive relation. It can be used to:

3. The numbers in the ____________ row of Pascal's Triangle provide the coefficients for the expansion of $(x + 1)^n$.

4. The recursive relation that specifies Pascal's Triangle is:

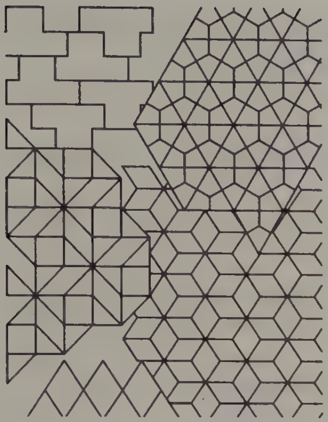
HOMEWORK

1. Use Pascal's Triangle to expand $(x + 1)^8$.

2. Use Pascal's Triangle to expand $(x + 1)^9$.

3. Refer to the Pascal's Triangle in the lesson. Shade in all of the odd numbers of the triangle. The resulting pattern is one that you've seen before.

LESSON 15: Pattern Collections



Introduction

Here you will have an opportunity to apply your knowledge of patterns to discussing and evaluating the pattern collections presented by members of your class.

Lesson Objectives
In this lesson, you will . . .
- apply the knowledge you have acquired about patterns to analyze and communicate information about patterns in the world

Material
- Each group will need to bring in its collection.

Pattern Collections

As each collection is presented, record the theme and the patterns exhibited in the collection.

Theme	Patterns Exhibited

Table 30 Pattern Collections

Other than your own, which collection was most interesting to you? Why?

Develop your own definition of the word *pattern.* Discuss and compare your definition with those of your group members.

Compare this definition with your definition from Lesson 1. Did anyone's definition change since then? Why?

Glossary

Butterfly Effect The fact that the choice of initiator can have a drastic effect on the outcome of the iteration.

Dodecahedron A polyhedron with 12 faces of regular pentagons; one of the Platonic solids.

Edge The segment where two faces of a polyhedron meet.

Face A surface of a polyhedron.

Fibonacci Sequence The Sequence of numbers 1, 1, 2, 3, 5, 8, 13, . . . where, starting with the third term, the next term is generated by adding the two previous terms.

Figurate numbers Numbers that can be represented by a polygon formed by dots.

Fractal A pattern created by performing a simple rule over and over.

Generator The pattern that shows the transformation to be performed on the previous output to create the next step.

Glide reflection A combination of a reflection and a translation along the line of reflection.

Golden Ratio If the measures of two lengths s and l satisfy $\frac{l}{s} = \frac{s + l}{l}$, then the ratio $\frac{l}{s}$ is the golden ratio and has a value of approximately 1.618033.

Hexagonal numbers Numbers that can be represented by a hexagon formed by dots: 1, 6, 15, 28, 45, ...

Hexahedron A polyhedron with six faces of squares; one of the Platonic solids.

Icosahedron A polyhedron with 20 faces of equilateral triangles; one of the Platonic solids.

Infinite detail A property of fractals where, theoretically, you can repeatedly zoom in on the image and see greater detail.

Initiator The starting point of an iteration process.

Iteration The process of applying a rule over and over again and using the previous output as input for the next stage.

Limit The result of iterating infinitely many times.

Octahedron A polyhedron with eight faces of equilateral triangles; one of the Platonic solids.

Pentagonal numbers Numbers that can be represented by a pentagon formed by dots: 1, 5, 12, 22, 35, ...

Polyhedron A three-dimensional object constructed from polygons.

Recursive relation One or more mathematical equations that specify a repetition of rules.

Reflection A rigid motion that creates a mirror image and is specified by a line of reflection.

Reflection symmetry Exhibited when a tile or tessellation can be folded in half along a midline and all edges match up.

Regular polygon A polygon with equal angles and equal lengths.

Regular polyhedron A three-dimensional object constructed from regular polygons.

Right angle An angle with a measure of 90°.

Rotation A rigid motion that turns the image around a center with a specified angle of rotation.

Rotation symmetry Exhibited when a tile or tessellation can be rotated a specified angle onto itself.

Self-similarity A property of fractals where the fractal contains smaller copies of itself.

Square numbers Numbers that can be represented by a square formed by dots: 1, 4, 9, 16, 25, 36, ...

Subscripts Written to the lower right of a variable, a subscript may specify the number of repetitions that are being applied to the variable.

Tetrahedron A polyhedron with four faces of equilateral triangles; one of the Platonic solids.

Tessellation A "tiling" made up of copies of a shape or shapes that fit together with no gaps or overlaps.

Tiles The building blocks of tessellations; the individual shape.

Translation A shift of a specified direction and distance.

Triangular numbers Numbers that can be represented by a triangle formed by dots: 1, 3, 6, 10, 15, 21, ...

Vertex The point on a polyhedron at which three or more faces meet.

Selected Answers

Lesson 1

none

Lesson 2

Homework
1. a. Yes b. Yes c. No d. No

Lesson 3

Table 6

Name	Number of Sides, n	Angle Measure, m	Angle Sum of the Polygon, s
Triangle	3	60°	180°
Square	4	90°	360°
Pentagon	5	108°	540°
Hexagon	6	120°	720°
Heptagon	7	≈128.57°	900°
Octagon	8	135°	1080°

Table 6 Angle Sums of Regular Polygons

Formula for the Angle Sum of a Polygon
$s = (n - 2) \cdot 180$

Homework
1.

Name	Number of Sides, n	Angle Measure, m	Angle Sum of the Polygon, s
Nonagon	9	140°	1260°
Decagon	10	144°	1440°
Dodecagon	12	150°	1800°
Pentadecagon	15	156°	2340°
Icosagon	20	162°	3240°

Table 8 Angle Sum of Larger Regular Polygons

Lesson 4

Homework
4. translation and reflection

Lesson 5

Activity 1: Table 11

Polygon	Number Meeting at Vertex	Angle Sum (<360°)
Equilateral triangle	3	180°
Equilateral triangle	4	240°
Equilateral triangle	5	300°
Square	3	270°
Pentagon	3	324°

Table 11 Vertex Angle Sum

Homework
1.

Name	Number of Faces	Polygon Used	Angle Sum of Vertex
Tetrahedron	4	triangle	180°
Hexahedron (cube)	6	square	270°
Octahedron	8	triangle	240°
Dodecahedron	12	pentagon	324°
Icosahedron	20	triangle	300°

Table 12 Polyhedra Summary

Lesson 6

Homework
2. They are listed in increasing order of the number of faces.
3. hexahedron, six seats

Lesson 7

Table 14

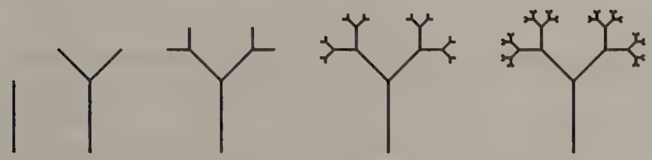
Name	Vertices	Edges	Faces
Tetrahedron	4	6	4
Cube	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12
Icosahedron	12	30	20

Table 14 Characteristics of the Platonic Solids

Euler–Descartes Formula
$V - E + F = 2$

Lesson 8

Fractal Tree



Lesson 9

Activity 1

1. $B_1 = 2$ $B_2 = 1.5$ $B_3 = 1.25$
$B_4 = 1.125$ $B_5 = 1.0625$ $B_6 = 1.03125$
$B_7 = 1.015625$ $B_8 = 1.0078125$ $B_9 = 1.00390625$
$B_{10} = 1.001953125$

3. $B_1 = 4$ $B_2 = 2.5$ $B_3 = 1.75$
$B_4 = 1.375$ $B_5 = 1.1875$ $B_6 = 1.09375$
$B_7 = 1.046875$ $B_8 = 1.0234375$ $B_9 = 1.01171875$
$B_{10} = 1.005859375$

Homework

1. $a_n = a_{n-1} +$ $a_1 = 2$ 2, 4, 6, 8, 10
3. $a_1 = 1$ No limit; 1, 2, 8, 128, increases without bound
$a_1 = 0.4$ Limit = 0
$a_1 = 2$ No limit; 8, 128, 32768, increases without bound

Lesson 10

Relationship Between Square and Triangular Numbers
$T_{n-1} + T_n = S_n$

Homework

2. $T_{10} = 55$
4.a.
$1^2 = 1$
$11^2 = 121$
$111^2 = 12321$
$1111^2 = 1234321$
5.a. 8

Lesson 11

Homework

2. No, $T_2 = 3$ is not a hexagonal number.

Lesson 12

Table 24

*n*th Previous Generation	Number of Male Bees in the *n*th Generation	*n*th Previous Generation	Number of Male Bees in the *n*th Generation
Current	1	6th	13
1st	1	7th	21
2nd	2	8th	34
3rd	3	9th	55
4th	5	10th	89
5th	8	11th	144

Table 24 Number of Male Bees

Homework

2. Looking at the female bees in each branch, we get Fibonacci Numbers as well.

Lesson 13

Table 29

Ratio of $\frac{F_{n+1}}{F_n}$	In Decimals	Ratio of $\frac{F_{n+1}}{F_n}$	In Decimals
$\frac{F_2}{F_1} = \frac{1}{1}$	1	$\frac{F_7}{F_6} = \frac{13}{8}$	1.625
$\frac{F_3}{F_2} = \frac{2}{1}$	2	$\frac{F_8}{F_7} = \frac{21}{13}$	1.615385 . . .
$\frac{F_4}{F_3} = \frac{3}{2}$	1.5	$\frac{F_9}{F_8} = \frac{34}{21}$	1.619048 . . .
$\frac{F_5}{F_4} = \frac{5}{3}$	1.666666	$\frac{F_{10}}{F_9} = \frac{55}{34}$	1.617647 . . .
$\frac{F_6}{F_5} = \frac{8}{5}$	1.6		

Table 29 Fibonacci Ratios

Lesson 14

Expansions

$(x + 1)^0 = 1$

$(x + 1)^1 = x + 1$

$(x + 1)^2 = x^2 + 2x + 1$

$(x + 1)^3 = x^3 + 3x^2 + 3x + 1$

$(x + 1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$

$(x + 1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

$(x + 1)^6 = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$

$(x + 1)^7 = x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$

Lesson 15

none

Index

The Maricopa Mathematics Modules support teaching and learning by providing real-world problems and activities enhancing classroom learning. This discovery-based learning program puts math in action by exploring key topics through intriguing activities.

This project contains several modules:

- Data and Graphs
- Geometry
- Linear Behavior
- Finance
- Probability
- Beat Ratios and Juggling Proportions
- Exponential Growth and Decay
- Non-Linear Behavior
- Representations of Data
- Sets and Logic
- Patterns
- Right Triangle Trigonometry
- Sampling
- Systems
- Functions

This project contains resources to support both instructors and students:

Instructor's Annotated Edition (IAE)
This is an exact replica of the student text but also features margin annotations with teaching tips along with answers to the exercises. A page of notes to the instructor precedes every lesson.

Instructor's Resource Manual with Solutions
This useful guide contains detailed notes on specific modules; notes on managing this type of course, and test questions. The solutions are complete for all exercises. (ISBN 0-618-000496)

Student Solutions Manual
This ancillary contains complete solutions to those exercises whose answers are found at the end of each module. (ISBN 0-618-00050X)

Look for these supplements in your bookstore or call Customer Service at 1.800.225.1464.

Visit us on the Web at www.hmco.com/college

HOUGHTON MIFFLIN
New Ways to Know

0-618-20639-6
90000
9 780618 206391
N00546